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PADÉ-TYPE APPROXIMATION TO FOURIER SERIES

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Contents

Preface

The subject of this book is Padé-type Approximation to Fourier series.

The text begins with definition and properties of Padé and Padé-type approximation to analytic functions. Padé approximants are rational functions whose expansion in ascending powers of the variable coincides with the Taylor power series expansion of analytic functions into a disk as far as possible, that is up to the sum of the degrees of the numerator and denominator. The numerator and denominator of a Padé approximant are completely determined by this condition and no freedom is left. Padé-type approximants are rational functions with an arbitrary denominator, whose numerator is determined by the condition that the expansion of the Padé-type approximant matches the Taylor power series expansions of analytic functions into a disk as far as possible, that is up to the degree of the numerator. The great advantage of Padé-type approximants over Padé approximants lies in the free choice of the poles which may lead to a better approximation.

One would like to adapt the proofs of one variable to the several variable case, however major obstacles and cumbersome formulas present themselves, making the applications almost unattainable. Indeed, many arguments in one variable use the Taylor power series expansion of analytic functions into the open disks. In several variables, the open polydisks do not enjoy a very elevated status and the domains of convergence of the power series representations exhibit a much greater variety than in one variable. In other words, the polydisk does not qualify to be the general target domain because of the failure of the property to be the maximal domain of convergence of a multiple power series. On the other hand, if $n>1$, the ring $\mathbf{P}(\mathbf{C}^n)$ of complex analytic polynomials in $\mathbf{C}^n$ is not principal and henceforth it is not an Euclidean ring. This means that when $n>1$ there is no division process in $\mathbf{P}(\mathbf{C}^n)$, which, in particular, implies that the cherished notion of continued fraction is absent from the theory of functions of several complex variables. Furthermore, in contrast to the one variable setting, there is no facility in the management of a logical connection between two apparently related mathematical entities: the polynomial of $\mathbb{C}^n$ and its degree. Finally, the singularities of rational functions of two or more complex variables are never isolated.

Since, because of all these reasons, many of the most highly appreciated theorems on rational approximation have no obvious analogue in several complex variables, one might expect that the theory of Padé and Padé-type approximants in $\mathbb{C}^n$ lacks the appeal of the classical one variable theory.

However, since 1976, there has been a great deal of work to determine the correct analogue and the properties of Padé and Padé-type approximants in dimensions exceeding two. In 1978, C. Brezinski was able to formulate the first definition of Padé-type approximants for double series, and from that time most other definitions dealing with homogeneous sub-expressions in a series extension were given. The papers "*Padé-type approximants in multivariables*" (by S. Arioka in Appl.Numer.Math.3(1987)497-511), "*Padé-type approximants for double power series*" (by C. Brezinski in J. Indian Math.Soc.42(1978)267-282), "*Padé and Padé-type approximants in several variables*"(by S. Kida in Appl.Numer.Math.6(1989/90)371-391) and "*A new family of Padé-type approximants in* $\mathbb{R}^k$" (by P. Sablonnière in J.Comput.Appl.Math.9(1983)347-359) give an overview of some of this work. In the paper "*Multivariate partial Newton-Padé and Newton-Padé-type approximants*" (by J. Abouir and A. Cuyt in J. Approx. Theory 72(1993)301-316), a general order definition was introduced that contained all the previous ones as special cases and was inspired on the definition of a general order multivariate Padé approximant as given by A. Cuyt in "*Multivariate Padé approximants revisited*" (BIT 26(1986)71-79). In all these definitions, the corresponding multivariate Padé-type approximation theory is based on Taylor series expansions on polydisks and leads to extremely complicated computations, at least for $n > 2$.

It would be reasonable to guess that the outlet lies with the consideration of another type of series representation for analytic functions. So, the principal aim of the present book is to propose a generalization of Padé and Padé-type approximation theory in one and several variables and to show how the proofs of a multidimensional Padé and Padé-type approximation theory can be cleared of their dependence on Taylor series and reconnected to original ideas of rational approximation by means of the Fourier series theory.

For this purpose, in *Chapter 1*, we shall first recall basic facts on rational approximation to the Taylor series representation of analytic functions. Then, by using the fundamental property of real harmonic functions to be real parts of analytic functions, we will define Padé and Padé-type approximants to the Fourier series expansion of a real-valued function u that is harmonic in the open unit disk D. These approximants are real parts of rational functions of type $(m, m+1)$, that is, of rational functions with numerator of degree at most m and denominator with degree at most $m+1$. The crucial property is that the numerator and denominator of a Padé approximant

$$\mathrm{Re}[m/m+1]_u$$

are uniquely determined by the condition that the Fourier series expansion of its restriction to any circle C_r of radius $r<1$ matches the Fourier series expansion of the restriction of u to C_r up to the $\pm(2m+1)^{th}$ – order's Fourier term. On the other hand, the numerator of a Padé-type approximant

$$\mathrm{Re}(m/m+1)_u$$

is determined by the condition that the Fourier series expansion of its restriction to any circle C_r of radius $r<1$ matches the Fourier series expansion of the restriction of u to C_r up to the $\pm m^{th}$ – order's Fourier term. Several numerical examples showing the efficiency of these approximants will be given, and convergence results will be established.

Next, we shall construct Padé and Padé-type approximants to the Fourier series representation of a complex-valued harmonic function in the open unit disk. The construction follows from a coordinate procedure, named composed approximation, and generalizes classical rational approximation to analytic functions, in the sense that any classical Padé-type or Padé approximant to an analytic function on the unit disk coincides with a composed Padé-type or Padé approximant to this function.

With this background, we shall be in position to use the solution of the Dirichlet problem in order to discuss the numerical evaluation of a $2\pi-$periodic L^p-function f on $[-\pi,\pi]$, or on the unit circle C, by using (composed) Padé-type approximants. The idea is to take (composed) Padé-type approximation to the Poisson integral of the periodic function, and then consider radial limits to approximate its Fourier series representation. Several numerical examples will confirm the expectation that these radial limits –named (composed) Padé-type approximants to the periodic function– may lead to satisfactory approximations. The fundamental result is that the Fourier series expansion of a (composed) Padé-type approximant to f matches the Fourier series expansion of f up to the $\pm m^{th}-$order's Fourier term.

The problem of convergence for a sequence of (composed) Padé-type approximants to a $2\pi-$periodic L^p-function is of considerable interest, and will therefore be extensively studied. Especially, for $p=2$, we shall describe how this problem of convergence is connected with Schur and Szegö's classical theories on orthogonal polynomials.

As an application, we shall see how Padé-type approximants to continuous $2\pi-$periodic real-valued functions may accelerate the convergence of functional sequences. More precisely, we shall investigate the assumptions under which, for every sequence of functions converging to a real-valued continuous $2\pi-$periodic function on $[-\pi,\pi]$, there is always a Padé-type approximation sequence converging point-wise to that function faster than the first sequence. This property permits us to construct better and better approximations to continuous functions.

Finally, by using Padé-type approximants, we shall propose an alternative and direct method for the numerical computation of derivatives and definite integrals.

In *Chapter 2*, we shall first consider interpolation methods for the numerical evaluation of a $2\pi-$periodic finite Baire measure μ on $[-\pi,\pi]$ or on the unit circle C. The idea will again be to take (composed) Padé-type approximants to the Poisson integral of μ, and then consider radial limits –the so-called (composed) Padé-type approximants to $\mu-$ to approximate the Fourier series representation of μ. The main property is that the Fourier series expansion of a (composed) Padé-type approximant to μ matches the Fourier series expansion of μ up to the $\pm m^{th}-$order's Fourier term.

Evidently, a serious and strong criterion for the successful application of such an interpolation method is determined by the convergence behavior of the corresponding (composed) Padé-type approximation sequence to the $2\pi-$periodic finite Baire measure. So, the next main purpose of this *Chapter* will be to prove concrete convergence results confirming the computational efficiency of these interpolation methods.

We shall also obtain integral representation formulas for (composed) Padé-type approximants to the Fourier series expansion of harmonic, analytic or L^p-functions, and, in this connection, we shall define and study (composed) Padé-type operators on the spaces of harmonic, analytic or L^p-functions. Application of these operators will furnish additional theoretical convergence results.

Chapter 3 is devoted to multidimensional and abstract Padé-type approximation.

As in the one variable case, for complex dimensions greater than one, any rational approximant depends on the choice of polynomials interpolating the Cauchy kernel. However, from the point of view of integral representations, a major difference between both cases is displayed. The difference is due to the fact that, in the one variable setting, there is essentially only one kernel –the Cauchy kernel– , while, in several variables, one has great freedom to modify, by a basically algebraic procedure, the original potential theoretic kernels.

In 1950, S. Bergman introduced new integral representation for analytic functions. The roots of his representation are based on abstract Hilbert space theory. The relevant abstract kernel –the so-called Bergman kernel– can be defined quite easily for arbitrary domains, but it is difficult to obtain concrete representations for it, except in special cases.

The first aim of *Chapter 3* will be to define Padé-type approximation to functions that are analytic and of class L^2 in Ω, by interpolating the Bergman kernel function

$$K_\Omega(z,x) = K_\Omega(z_1,\dots,z_n,x_1,\dots,x_n)$$

into an arbitrary open bounded open set Ω in $\mathbb{C}^n$, instead of the Cauchy kernel

$$(1-x_1z_1)^{-1}\dots(1-x_nz_n)^{-1}$$

into an open polydisk centered at the origin.

The Bergman kernel function $K_\Omega(z,x)$ belongs to the Hilbert space

$$O\ L^2(\Omega)$$

of all functions that are analytic and of class L^2 in Ω. For any orthonormal basis

$$\{\varphi_j : j = 0,1,\dots\}$$

of $O\ L^2(\Omega)$, the idea will be to replace the kernel $K_\Omega(z,x)$ by simpler interpolating expressions consisting of generalized polynomials

$$g_m(x,z) = \sum_{j=0}^{m} c_j^{(m)}(z)\,\overline{\varphi_j(x)},$$

such that

$$g_m(\pi_{m,k}, z) = K_\Omega(\pi_{m,k}, z)$$

for any $\pi_{m,k}$ in a finite set of pair-wise distinct points

$$\mathrm{M}_{m+1} = \{\pi_{m,0}, \pi_{m,1}, \dots, \pi_{m,m}\} \subset \Omega$$

with

$$\mathrm{M}_{m+1} \cap \bigcup_{0 \le j \le m} Ker\overline{\varphi_j} = \emptyset.$$

Then, by using appropriate approximate formulas, we shall define generalized Padé-type approximation to any $f \in O\ L^2(\Omega)$: the function

$$\sum_{j=0}^{m} a_j^{(f)} c_j^{(m)}(z) \in O\ L^2(\Omega)$$

will be a generalized Padé-type approximant to f, with generating system

$$\mathrm{M}_{m+1} = \{\pi_{m,0}, \pi_{m,1}, \dots, \pi_{m,m}\}.$$

Here, $a_j^{(f)}$ is the $j^{th}-$order's Fourier coefficient of f with respect to the basis $\{\varphi_j : j = 0,1,2,\dots\}$:

$$a_j^{(f)} = \int_\Omega f\ \overline{\varphi_j}\ dV.$$

Indicative numerical examples involving generalized Padé-type approximants to analytic L^2-functions of two complex variables will be produced to show the efficiency of these approximations.

As we shall see, under certain strong algebraic presuppositions on the generating system choice, one can also define Padé-type approximation to the function f. In analogy with the one variable setting, the crucial property is that the Fourier series expansion of a Padé-type approximant to f matches the Fourier series expansion of f up to the $\pm m^{th}-$order's Fourier term. In fact, as we shall show, if

$$\sum_{j=0}^{\infty} b_j^{(m,f)} \varphi_j(z)$$

is the Fourier expansion of the Padé-type approximant to $f \in O\ L^2(\Omega)$ with respect to the basis

$$\{\varphi_j : j = 0,1,2,\ldots\},$$

then

$$b_j^{(m,f)} = a_j^{(f)} \text{ for any } j = 0,1,2,\ldots,m .$$

We shall also consider the natural extension of these ideas, based on abstract Hilbert space theory, to the context of continuous functions on a compact subset of $\mathbb{R}^n$, and, more generally, to the elements of an arbitrary functional Hilbert space.

To do so, we shall first prove that a $C^{\infty}-$function f on a compact subset E of $\mathbb{R}^n$ satisfying Markov's classical inequality (M_{∞}), or Markov's inequality (M_2) with respect to some positive measure, has a Fourier series representation with respect to a Schauder basis consisting of orthogonal polynomials. This will permit us to give the definition of a generalized Padé-type approximant to the Fourier series representation of f. Error and global convergence results for the asymptotic behaviour of a sequence of generalized Padé-type approximants to the Fourier series of f will be demonstrated. Further, under certain strong algebraic presuppositions on the generating system's choice, one will can also define Padé-type approximants to the function f. The crucial property will again be that the Fourier series expansion of a Padé-type approximant to f matches the Fourier series expansion of f up to the $\pm m^{th}-$order's Fourier term.

Next, we shall propose an extension of these ideas into every functional Hilbert space H consisting of functions defined into an arbitrary topological space Χ and with values into the extended complex plane. Let $\langle \cdot / \cdot \rangle_H$ be the inner product of H. For any complete self-summable orthonormal family

$$\mathrm{N} = \{e_j : j = 1,2,\ldots\}$$

in H, the function

$$K_{\mathrm{X}}(z,\cdot) : \mathrm{X} \to \mathbb{C} : x \mapsto K_{\mathrm{X}}(z,x) = \sum_{j=0}^{\infty} e_j(z)\overline{e_j(x)}$$

belongs to H ($z \in \mathrm{X}$), and each $u \in H$ has the Fourier expansion

$$u(x) = \sum_{j=0}^{\infty} \langle u / e_j \rangle_H \, e_j(z) \qquad (z \in \mathrm{X}).$$

Since

$$u(z) = T_u(K_{\mathrm{X}}(z,x)),$$

where T_u is the linear functional $T_u : \overline{E} \to \mathbb{C}$ defined on the complex vector space $\overline{E}$ that is generated by all finite complex combinations of $\overline{e_j}$'s by

$$T_u\left(\overline{e_j(x)}\right) := \langle u / e_j \rangle_H ,$$

the function $K_X(z,x)$ has to be replaced by a simpler expression

$$G_m(x,z) = \sum_{j=0}^{m} \sigma_j^{(m)}(z)\overline{e_j(x)},$$

fulfilling

$$G_m(x,z) = K_X(z,\pi_{m,k}) \qquad \text{for any } k \le m$$

in a finite set of pair-wise distinct points

$$\mathrm{M}_{m+1} = \{\pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}\} \subset X - \left(\bigcup_{0 \le j \le m} Ker\overline{e_j} \right)$$

($Ker\overline{e_j}$ is the kernel of $\overline{e_j}$). Any function

$$T_u(G_m(x,z)) = \sum_{j=0}^{m} \langle u / e_j \rangle_H \, \sigma_j^{(m)}(z)$$

is a generalized Padé-type approximant of $u \in H$ with generating system M_{m+1}.

As we shall show, under certain strong algebraic presuppositions on the generating system choice, one can also define Padé-type approximation to the function f. The characteristic property of such an approximation is that the Fourier expansion of a Padé-type approximant to u matches the Fourier expansion of u up to the $\pm m^{th}$ − order's Fourier term, in the sense that

$$< T_u(G_m(x,\cdot)) / e_j >_H = < u / e_j >_H , \qquad \textit{for any } j = 0,1,\ldots,m .$$

Finally, after introducing representation for the generalized Padé-type approximants to elements of H, we shall conclude with the definition and convergence investigation of generalized Padé-type approximation to any linear operator

$$H \to H .$$

As an application, we will then use generalized Padé-type approximants to the Bergman projection operator to prove an extension of Painlevé' s classical Theorem on the continuous extension of analytic functions to the boundary of an arbitrary bounded open subset of $\mathbb{C}^n$. Finally, we shall consider some indicative numerical examples involving generalized Padé-type approximants.

It gives me great pleasure to express my gratitude to the two persons who have had the most significant and lasting impact on my training as a mathematician. First, I want to mention Claude Brezinski. His important work on rational approximation introduced me to the subject and provided the stimulus to study it further. His early support and his continued interest in my mathematical development, even after I left the University of Sciences and Techniques of Lille in France, is deeply appreciated. I discussed my plans for this book with him, and his encouragement contributed to getting the project started. Once I came to the France, I was fortunate to study under Gerard Cœuré. His lectures, which I was privileged to hear while a student at the University of Sciences and Techniques of Lille, introduced me to the theory of Several Complex Variables, a fertile ground for applying the new tools of representations, and supervised my dissertation.

Finally, I want to express my deepest appreciation to my family, who, for the past few years, had to share me with this project. Without the constant encouragement and understanding of my wife Kalliopi and my son Ioannis, it would have been difficult to bring this work to completion.

Nicholas J. Daras

Chapter 1

On the Numerical Evaluation of Harmonic and 2π-Periodic L^p-Functions by a few of their Fourier Coefficients

Summary

The numerical evaluation of a harmonic or 2π-periodic L^p-function by its Fourier series representation may become a difficult task whenever only a few coefficients of this series expansion are known or it converges too slowly. In this *Chapter*, we will propose a general method to evaluate such any function by means of composed Padé and Padé-type approximants. The definition and properties of these rational approximants will be given. After having done this successfully, we will consider several concrete examples and will give theoretical applications to the convergence acceleration problem of functional sequences. Finally, an alternative method for the numerical computation of derivatives and definite integrals will be defined.

Introduction

In this *Chapter* we present Padé and Padé-type approximation to harmonic or $2\pi-$periodic L^p-functions. The text begins with a review of standard local results, followed by a discussion of classical concepts on *rational approximation* related to the extension properties of analytic functions in one complex variable. It then continues with a natural generalization to the context of harmonic and L^p-functions, and concludes with several applications.

Padé-type approximation is the rational-function analogue of the Taylor polynomial approximation to a power series. Before proceeding to a formal definition, let us recall a few facts from elementary calculus and interpret these as results in approximation theory.

One of the strong motivations for studying rational approximations is the perennial and concrete problem of representing functions efficiently by easily computed expressions. In this capacity the rational functions

$$R(x) = \frac{a_0 + a_1 x + \ldots + a_n x^n}{b_0 + b_1 x + \ldots + b_m x^m}$$

have been found to be extremely effective. In a loose manner of speaking, one may say that the curve-fitting ability of $R(x)$ is roughly equal to that of a polynomial of degree $n+m$. However, we shall see that in competing with the polynomial of degree $n+m$, $R(x)$ has an unsuspected advantage in that the computation of $R(x)$ for a given x does not require $n+m$ additions, $n+m-1$ multiplications, and one division as might be surmised at first. By transforming $R(x)$ into a continued fraction

$$(CF) \qquad R(x) = P_1(x) + \cfrac{c_2}{P_2(x) + \cfrac{c_3}{P_3(x) + \ddots + \cfrac{c_k}{P_k(x)}}}$$

(in which each P_j denotes a certain polynomial), we achieve the significant reduction in the number of *«long»* arithmetic operations (multiplications and divisions) to n or m.

Theorem. *Any rational function $R(x)$ can be put into the continued fraction form (CF), and from this it can be evaluated for any x with at most $\max\{n,m\}$ long operations.*

Proof. Let the numerator and denominator be denoted by R_0 and R_1, respectively. Let $\deg(\cdot)$ stand for *degree of*, and assume first that $\deg(R_0) \geq \deg(R_1)$. By successive division (of R_{j-1} by R_j) we obtain quotients Q_j and remainders R_{j+1} as follows:

$$R_0 = R_1 Q_1 + R_2 \quad (\deg(R_2) < \deg(R_1)), \quad R_1 = R_2 Q_2 + R_3 \quad (\deg(R_3) < \deg(R_2)), \text{ etc.}$$

Since the degrees $\deg(R_j)$ form a decreasing sequence of nonnegative integers, we eventually reach a step in which $\deg(R_k) = 0$:

$$R_{k-2} = R_{k-1} Q_{k-1} + R_k \quad (\deg(R_k) = 0) \quad \text{and} \quad R_{k-1} = R_k Q_k \; .$$

From this schema, we have

$$R = \frac{R_0}{R_1} = Q_1 + \frac{1}{R_1/R_2} = \ldots = Q_1 + \cfrac{1}{Q_2 + 1\Big/ Q_3 + \ddots + 1\Big/ Q_{k-1} + 1\Big/ Q_k} \, .$$

This can also be written in the equivalent form (CF), each P_j except P_1 being a monic polynomial (i.e., P_j has leading coefficient unity). The numerical evaluation of such a polynomial requires no more than $\deg(P) - 1$ multiplications, since it can be expressed in the form

$$x^{\nu} + A_{\nu-1} x^{\nu-1} + \ldots + A_1 x + A_0 = (\ldots((x + A_{\nu-1})x + A_{\nu-2})\ldots)x + A_0 \; .$$

The long operations necessary to calculate $R(x)$ from equation (CF) are then the multiplications for P_j and $k-1$ divisions. The total number of these operations is

$$\begin{aligned} \deg(P_1) + \deg(P_2) + \ldots + \deg(P_k) &= \deg(Q_1) + \deg(Q_2) + \ldots + \deg(Q_k) \\ &= [\deg(R_0) - \deg(R_1)] + [\deg(R_1) - \deg(R_2)] + \ldots + [\deg(R_{k-1}) - \deg(R_k)]\mathrm{H} \\ &= \deg(R_0) \leq n \, . \end{aligned}$$

ere we have used the inequalities $\deg(R_{j+1}) < \deg(R_j)$ to conclude that

$\deg(R_j) = \deg(R_{j+1}) + \deg(Q_{j+1})$. Note that if Q_1 is monic, then the number of operations is at most $n-1$.

Now, in the case that $\deg(R_0) < \deg(R_1)$, we write

$$R = \frac{c_1}{c_1 R_1 / R_0},$$

where c_1 is selected so that $c_1 R_1$ and R_0 have the same leading coefficient. The preceding discussion now shows that $c_1 R_1 / R_0$ may be expressed as a continued fraction, any value of which may be computed with no more than $\deg(R_1) - 1$ long operations. Hence, in this case the evaluation of R requires at most $\deg(R_1) \leq m$ long operations.

A few historical comments might now help to put matters in perspective. Our principal sources of information are Brezinski's precious papers: *The long history of continued fractions and Padé approximants* (in "Padé approximation and its applications. Amsterdam 1980", M.G. de Bruin and H. Van Rossum eds., Lectures Notes in Mathematics 888, Springer Verlag, Heidelberg, 1981), and *The birth and early developments of Padé approximants* (presented at the 86th summer meeting of the American Mathematical Society, Toronto, August 23-27, 1982).

The first use of continued fractions goes back to the algorithm of Euclid (c.306 B.C.-c. 283 B.C.) for computing the g. c. d. of two positive integers which leads to a terminating continued fraction. Euclid's algorithm is related to the approximate simplification of ratios as it was practiced by Archimedes (287 B.C.-212 B.C.) and Aristarchus of Samos (c.310 B.C.-c.230 B.C.). Continued fractions were also implicitly used by Greek mathematicians, such as Theon of Alexandria (c.365 B. C.), in methods for computing the side of a square with a given area. Another very ancient problem which also leads to the early use of continued fractions is the problem of the diophantine equations in honor to Diophantus (c.250 A.D.) who found a rational solution of the equation $ax \pm by = c$, where a, b and c are given positive integers. This problem has been completely solved by the Indian mathematician Aryabhata (475-550), who wrote down explicitly the continued fraction for a/b. Around 1150, one of the most important Indian mathematicians, Bhascara, wrote a book "*Līlāvatī*", where he treated the equation $ax - by = c$. He proved that the solution can be obtained from the continued fraction for a/b. He also showed that the convergents $C_k = A_k / B_k$ of this continued fraction satisfy:

$$A_k = q_k A_{k-1} + A_{k-2},\ B_k = q_k B_{k-1} + B_{k-2} \quad \text{and} \quad A_k B_{k-1} - A_{k-1} B_k = (-1)^{k-1}.$$

Then the solution is given by $x = \mp c B_{n-1} + bt$ and $y = \mp c A_{n-1} + at$, according as a $B_{n-1} - b A_{n-1} = \pm 1$.

In Europe, the birth place of continued fractions is the north of Italy. The first attempt for a general definition of a continued fraction was made by Leonardo Fibonacci (c.1170-c. 1250). In his book "*Liber Abaci*" (written in 1202, revised in 1228 but only published in 1857), he introduced a kind of ascending continued fraction which is not of great interest. The first mathematician who really used our modern infinite continued fractions was Rafael Bombielli (1526-1572) the discoverer of imaginary numbers. In his book "*L' Algebra Opera*", published in 1579 in Bologna, he gave a recursive algorithm for extracting the square root of 13 which is completely equivalent to the infinite continued fraction

$$\sqrt{13} = 3 + \frac{4|}{|6} + \frac{4|}{|6} + \ldots$$

(The notation $b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ldots$ for the continued fraction

$$b_0 + {a_1}\Big/{b_1 + {a_2}\big/{b_2 + \ddots}}.$$

has been introduced in 1898 by Alfred Pringsheim (1850-1941)). The next and most important contribution to the theory of continued fractions is by Pietro Antonio Cataldi (1548-1626) who can be considered as the real founded of the theory. In his book "*Trattato del modo brevissimo di trovare la radice quadra delli numeri...*" published in Bologna in 1613, he followed the same method as Bombielli for extracting the square root and he was the first to introduce a symbolism for continued fractions. He computed the continued fraction for $\sqrt{18}$ up to the 15th convergent and proved that the convergents are alternatively greater and smaller than $\sqrt{18}$ and that they converge to it. The words "*continued fractions*" were invented in 1655, by the English mathematician John Wallis (1616-1703), in his book "*Arithmetica Infinitorum*", where the author gave for the first time our modern recurrence relationship for the convergents of a continued fraction. We also mention the Dutch mathematician and astronomer Christiaan Huygens (1629-1695) who built, in 1682, an automatic planetarium. He used continued fractions for this purpose as described in his book "*Descriptio automati planetari*" published after his death.

The major contribution to the theory of continued fractions is due to Leonhard Euler (1707-1783). In his first paper on the subject, dated 1737, he proved that every rational number can be developed into a terminating continued fraction, that an irrational number gives rise to an infinite continued fraction and that a periodic continued fraction is the root of a quadratic equation. He also gave the continued fractions for $e, \frac{e+1}{e-1}, \frac{e-1}{2}$ by integrating the Riccati equation by two different methods. Apart from the convergence of these continued fractions which he did not treated, Euler proved the irrationality of e and e^2. Euler's celebrated book "*Introductio in analysis infinitorum*", published in Lausanne in 1748, contains the first extensive and systematic exposition of the theory of continued fractions. In chapter 18, he gives the recurrence relationship for the convergents $C_k = \frac{A_k}{B_k}$ of the continued fraction $b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \ldots$ and then shows how to transform a continued fraction into a series

$$C_n - C_{n-1} = (-1)^{n-1} \frac{a_1 \ldots a_n}{B_1 B_n}.$$

This leads to the relation

$$C = b_0 + \sum_{n=1}^{\infty} (-1)^n \frac{a_1 \ldots a_n}{B_{n-1} B_n}.$$

Reciprocally, Euler shows that an infinite series can be transformed into a continued fraction

$$\sum_{n=1}^{\infty} (-1)^{n-1} C_n = \cfrac{C_1}{1 + \cfrac{C_2}{C_1 - C_2 + \ddots + \cfrac{C_{n-2} C_n}{C_{n-1} - C_n + \ldots}}}$$

After some examples, he treats the case of a power series. Then, he comes to the problem of convergence showing how to compute the value of the periodic continued fraction $C = \frac{1|}{|2} + \frac{1|}{|2} + \ldots$ by writing $C = \frac{1}{2+C}$ which gives $C^2 + 2C = 1$ and thus $C = \sqrt{2} - 1$. From this example he derives Bombielli' s method for the continued fraction expansion of the square root and a general method for the solution of a quadratic equation. The chapter ends with Euclid's algorithm and the simplification of fractions with examples.

Euler published some papers where he applied continued fractions to the solution of Riccatti' s differential equation and to the calculation of integrals. He also showed that certain

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continued fractions derived from power series can converge outside the domain of convergence of the series. In a letter dated 1743 and in a paper published in 1762, Euler investigated the problem of finding the integers a for which a^2+1 is divisible by a given prime of the form $4n+1=p^2+q^2$. Its solution involves the penultimate convergence of the continued fraction for $\frac{p}{q}$. In 1765, Euler studied the Pellian equation $x^2=Dy^2+1$. He developed $\sqrt{D}$ into a continued fraction. In 1771, Euler applied continued fractions to the approximate determination of the geometric mean of two numbers whose ratio is as $\frac{1}{x}$. The method can be used to get approximate values of $x^{p/q}$. In 1773, Euler used continued fractions to find x and y making mx^2-ny^2 minimum, and in 1780 for seeking f and g such that $fr^2-gs^2=x$. In 1783, Euler proved that the value of the continued fraction

$$m+1\Big/2+m+2\Big/3+\ddots$$

is a rational number, when m is an integer not smaller than 2.

Thus, Euler was the first mathematician not only to give a clear exposition of continued fractions but also to use them extensively to solve various problems. He was quite familiar to the process of transforming a power series into a continued fraction. His method for performing this transformation is simply the division process which is quite similar to Euclid's algorithm for obtaining the g. c. d. of two positive integers. He used this technique in, at least, two papers (: one in 1775 and another in 1776), and he was led to use rational approximations to power series which are, in fact, Padé approximants. In a letter to Christian Goldbach (1690-1764), dated October 17, 1730, Euler considers the series

$$S(x) = x + \frac{1}{2\times 3}\frac{x^3}{b^2} + \frac{1\times 3}{2\times 4\times 5}\frac{x^5}{b^4} + \frac{1\times 3\times 5}{2\times 4\times 6\times 7}\frac{x^7}{b^6} + \ldots,$$

where b is the diameter of a circle, x is the chord and S the corresponding arc. He gives, without any explanations, the following approximations of $S(x)$

$$\frac{60b^2x - 17x^3}{60b^2 - 27x^2} \qquad \text{and} \qquad x + \frac{840b^2x^3 - 122x^5}{120b^2(42b^2 - 25x^2)}.$$

It is easy to check that the first approximation satisfies $S(x) + O(x^7)$ and thus is identified with the Padé approximant $[3/2]_S(x)$. The second approximation satisfies $S(x) + O(x^9)$ and thus is identified with $[5/2]_S(x)$. Another important source about Euler's work on rational approximation is its correspondence with the German astronomer Tobias Mayer (1723-1762). On July 27, 1751, Euler answered to Mayer's question on the solvability of the differential equation $dy = \frac{1}{\log x}dx$ by showing that the series

$$y(x) = -Ux[1 - 1\times U + 1\times 2\times U^2 - 1\times 2\times 3\times U^3 + \ldots + (-1)^\nu \nu!\times U^\nu + \ldots]$$

(with $\log x = U$) satisfies this equation. In order to determine the values of this series, Euler wrote that the series

$$S(U) = 1 - 1\times U + 1\times 2\times U^2 - 1\times 2\times 3\times U^3 + \ldots + (-1)^\nu \nu!\times U^\nu + \ldots$$

is equal to the following continuous fraction

$$S(U) = \cfrac{1}{1+\cfrac{U}{1+\cfrac{U}{1+\cfrac{2U}{1+\cfrac{2U}{1+\cfrac{3U}{1+\cfrac{3U}{1+\ddots}}}}}}}.$$

This fraction always closely determines the limits of $S(U)$'s value, and thus one can approximate to the value of $S(U)$ as closely as one will. Then the values approximating to $S(U)$ are:

$$1, \frac{1}{1+U}, \frac{1+U}{1+2U}, \frac{1+3U}{1+4U+2U^2}, \frac{1+5U+2U^2}{1+6U+6U^2}, \frac{1+8U+11U^2}{1+9U+18U^2+6U^3},$$
$$\frac{1+11U+26U^2+6U^3}{1+12U+36U^2+24U^3}, \ldots$$

of which every alternate one is greater than S. It is easy to check that the rational fractions given by Euler are the Padé approximants $[0/0]_S(U), [0/1]_S(U), [1/1]_S(U), [1/2]_S(U), [2/2]_S(U), [2/3]_S(U), [3/3]_S(U)$, etc.

It must be noticed that Padé approximants can also be found in a letter, dated September 16/27, 1740, of an unknown English mathematician Georges Anderson to William Jones (1675-1749), where Anderson considered Padé approximants to $\log(1+x)$ and he went one step farther that Euler since he gave the first term of the error. About the same time, Daniel Bernoulli (1700-1782) used similar rational fractions in order to invert the power series

$$y = x + ax^2 + bx^3 + \ldots$$

He wants to express x in terms of y. He first writes x as a power series in y. The method of indeterminate coefficients gives

$$x = y - ay^2 + (2a^2 - b)y^3 + \dots$$

On the other hand, one has

$$1 = \frac{1}{y}x + \frac{a}{y}x^2 + \frac{b}{y}x^3 + \dots$$

By using his famous method of finding the smallest zero of an infinite power series (published in two memoirs in 1730) applied to difference equations of infinite order, he obtains the sequence

$$\dots,0,0,1,\frac{1}{y},\frac{1}{y^2}+\frac{a}{y},\frac{1}{y^3}+\frac{2a}{y^2}+\frac{b}{y},\dots$$

The ratio of two consecutive terms of this sequence gives an approximate value for x. For example, he has

$$x = \frac{y + 3ay^2 + (a^2 + 2b)y^3 + cy^4}{1 + 4ay + 3(a^2 + b)y^2 + (2ab + c)y^3 + dy^4}.$$

Thus, x is approximated by a rational fraction in y. If this rational fraction is developed into an ascending power series in y (by effecting the division), it matches the series obtained from the indeterminate coefficients method up to the term whose degree equals the degree of the numerator. This kind of approximation is weaker than Padé approximation whose degree of approximation is equal to the sum of the degrees of the numerator and the denominator of the rational fraction. Such approximations are now called Padé-type approximations.

However, neither Euler nor Anderson and D. Bernoulli and Johann Heinrich Lambert (1728-1777) (who also gave a direct approach to Padé approximants in his paper "*Observationes* *variae in Mathesin puram*", published in 1758 in Acta Helvetica) can be credited with the discovery of Padé approximants (or of Padé-type approximants), since they were not aware of their fundamental property of matching the series up to the term whose degree is equal to the sum of the degrees of the numerator and of the denominator (or respectively, of their fundamental property of matching the series up to the term whose degree equals the degree of the numerator).

The first mathematician to be conscious of this property was Joseph Louis Lagrange (1736-1813) in a paper dated 1776 and dealing with the solution of differential equations by means of continued fractions. Transforming the convergents of these continued fractions into rational fractions by using their recurrence relationship he claims that they match the series "*up to the power of x inclusively which is the sum of the highest powers of x in the numerator and in the denominator*". As it is noticed by Brezinski, this paper is really the birth-certificate of Padé approximants.

Many other important contributions to the theory of continued fractions are due to Lagrange. In 1766, he gave the first proof that $x^2 = D y^2 + 1$ has integral solutions with $y \neq 0$, if D is a given positive integer not a square. The proof makes use of the continued fraction for $\sqrt{D}$. In 1767, Lagrange published a "*Mémoire sur la résolution des équations numériques*", where he gave a method for approximating the real roots of an equation by continued fractions. One year later he wrote an "*Addition*" to the preceding "*Mémoire*", where he proved the converse of Euler's result. He showed that the continued fraction for $\sqrt{D}$ is periodic and that the period can only take two different forms which he exhibited. He related his results to the solution of $x^2 = D y^2 \pm 1$. In the same paper he extended Huygens' method for solving $p y - q x = r$. An interesting problem treated by Lagrange in 1772 is the solution of linear difference equations with constant coefficients. In 1774, in an addition to Euler's Algebra, Lagrange proved that if a is a given positive real number then relatively prime integers p and q can be found such that $p - qa < r - sa$ for $r < p$ and $s < q$ by taking p/q as any convergent of the continued fraction for a in which all the terms are positive. He also gave a method, using continued fractions, to solve $A y^2 - 2B y z + C z^2 = 1$ in integers and he proved that Pell's equation cannot be solved by use of a continued fraction for $\sqrt{D}$ in which the signs of the partial denominators are arbitrarily chosen.

Following these predecessors many mathematicians of the nineteenth century became interested by continued fractions. All those who worked on the transformation of a formal power series into a continued fraction, by using for example the division process, have in fact obtained Padé approximants since, in most of the cases, the division process leads to the continued fraction corresponding to the power series whose successive convergents are

$$[0/0], [0/1], [1/1], [1/2], [2/2], \ldots$$

As the history of Padé approximants is very much interlaced with that of continued fractions we shall not follow that way and we shall only look now at the direct approaches to Padé approximants that do not make use of continued fractions. However we would like to mention one more contribution of that type since it opened a very important new chapter in mathematics. In his very celebrated paper on Gaussian quadrature methods, presented to the Göttingen Society on September 16, 1814, Carl Friedrich Gauss (1777-1855) proved that

$$\log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$$
$$= \cfrac{1}{x^{-1} - \cfrac{1/3}{x^{-1} - \cfrac{2\times 2/3\times 5}{x^{-1} - \cfrac{3\times 3/5\times 7}{x^{-1} - \ddots}}}}$$

The convergents of this continued fraction are the Padé approximants of the series. The denominators of the convergents are the Legendre orthogonal polynomials as proved by Pafnouty Lvovitch Tchebycheff (1821-1894).

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In a paper published in 1837, but dating from November 1834, Ernst Eduard Kummer (1810-1893) made use of Padé approximants for summing slowly convergent series. Kummer writes exactly the equations defining the $[n/n+1]$ Padé approximant, he gives several examples, but he does not prove any theoretical result.

In 1845, Carl Gustav Jacobi (1804-1851) proved his celebrated formula for Padé approximants. In the same paper, he gives several representations for the numerators and the denominators of Padé approximants, all derived as special cases of interpolating rational fractions studied by Augustin Louis Cauchy (1789-1857). Jacobi's representations are based on the systems of linear equations defining the Padé approximants.

Georg Friedrich Bernhard Riemann (1826-1866) proved in October 1863 the convergence of the corresponding continued fraction given by Gauss for the ratio of two hypergeometric series. The proof was found in Riemann's papers after his death. It uses integration in the complex domain, and it was completed and edited by Hermann Amandus Schwarz (1843-1921). According to Henri Eugène Padé (1863-1953), this is the first proof of convergence for Padé approximants.

In his thesis dated 1870, Georg Ferdinand Frobenius (1849-1917) showed that the numerators, the denominators and the errors of the convergents of the continued fraction

$$C(x) = \frac{1}{a_0 x - \frac{1}{a_1 x - \frac{1}{a_2 x - \ddots}}}$$

are related by three terms recurrence relationships. These results were extended in a paper published in 1881 where he gave the relations linking the numerators and the denominators of three adjacent approximants in the Padé table. Some of these identities, now known as the Frobenius identities, are connected with Jacobi's determinant formulas for the coefficients of the continued fraction

$$a_0 + \frac{x}{a_1 + \frac{x}{a_2 + \frac{x}{a_3 + \ddots}}}.$$

The successive convergents of this fraction form the main diagonal of the Padé table. A recursive method for computing $a_0, a_1, a_2, \ldots$ is given by Frobenius who, in fact, gave the first systematic study of Padé approximants and placed their theory on a rigorous basis.

Numerous contributions to Padé approximants are also due to Edmond Nicolas Laguerre (1843-1886). In his first paper of 1876, he treats the cases $(x^2-1)^{-1/2}, (x+a)^m/(x+b)^m$ and $e^{p(x))}$ where $p(x)$ is a polynomial. In his second note of 1876, he studies $\exp\left(Arc\tan x^{-1}\right)$ and in 1879, he works out the case of the series

$$\frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots$$

He shows the convergence of the sequence $\{[k/k] : k \in \mathbb{N}_0\}$ to $e^x \int_x^\infty e^{-t} t^{-1} dt$, and he also treats the case

$$\int_x^\infty e^{-t^2} dt .$$

In 1881, Leopold Krönecker (1823-1891) considered the problem of finding a rational fraction $p(x)/q(x)$ having the same derivative at a given point that a given function $f(x)$. He used two techniques for solving this problem. The first one is the Euclidean division algorithm for finding the continued fraction expansion of $g(x)/f(x)$. The second method is to solve the system of linear equations obtained by imposing that the first coefficients of the power series expansion of $f\,q - g\,p$ vanish.

At the same year (1881), in his Inaugural Dissertation, Karl Heun presented the connection between orthogonal polynomials, continued fractions and Padé approximants. Let $\{p_\nu(x): \nu \in \mathbb{N}_0\}$ be a family of orthogonal polynomials on the closed interval $[a,b]$ with respect to a measure $d\,\mu$ that is

$$\int_\alpha^\beta p_\nu(x) p_k(x) d\mu(x) = 0 \quad (\text{if } \nu \neq k).$$

These polynomials satisfy a tree terms recurrence relationship

$$p_\nu = (A_\nu x + B_\nu) p_{\nu-1}(x) - C_\nu p_{\nu-2}(x), \nu \geq 0.$$

Let us consider the continued fraction

$$1 \Big/ A_1 x + B_1 - C_2 \Big/ A_2 x + B_2 - C_3 \Big/ A_3 x + B_3 - C_4 \Big/ A_4 x + B_4 - \ddots.$$

The convergents $\dfrac{R_\nu(x)}{S_\nu(x)}$ of this fraction are the Padé approximants $[0/1]$, $[1/2]$,….Then

$$S_\nu(x) = \sqrt{C_0}\, p_\nu(x), \text{ where } C_k = \int_a^b x^k d\mu(x).$$

It has been proved in 1896 by Andrei Andrevitch Markov (1856-1922) that if x is an arbitrary point in the complex plane cut along $[a,b]$, then

$$\lim_{\nu\to\infty}\frac{R_\nu(x)}{S_\nu(x)}=\frac{1}{C_0^{\,2}}\sqrt{C_0C_2-C_1^{\,2}}\int_a^b\frac{d\mu(t)}{x-t},$$

and that the convergence is uniform on every compact set in the complex plane having no point in common with $[a,b]$. Markov's result is a consequence of *Stieltjes' s Theorem* on the convergence of Gaussian quadrature methods.

Another important contribution I would like to mention is that of Charles Hermite (1822-1901). The first reason for that choice is that he was Padé' s advisor, the second reason is that he defined the approximants which are now called the Padé -Hermite approximants. The third reason is that he proved the fundamental result that the number e is a transcendental number and that the proof used Padé approximants. Hermite' s *Proof* is a follows. e is assumed to be an algebraic number, that is satisfying $a_0+a_1e+...+a_ne^n=0$ for some integers $a_0,a_1,...,a_n$. Hermite looks for the polynomials $Q(x),P_0(x),...,P_n(x)$, of degree k, such that $e^{jx}Q(x)-P_j(x)=\mathrm{O}\left(x^{(n+1)k+1}\right)$ for $j=0,1,2,...,n$. Then

$$T(x)=\sum_{j=0}^{n}a_jP_j(x)-Q(x)\sum_{j=0}^{n}a_je^{jx}=\mathrm{O}(x^{(n+1)k+1}).$$

Since $|T(1)|<1$ and is an integer for k sufficiently large, it follows that

$$T(1)=\sum_{j=0}^{n}a_jP_j(1)=0.$$

Giving to k the values $k,k+1,...,k+n$, Hermite proves that $a_0+a_1e+...+a_ne^n\neq 0$ which contradicts the assumption. This last part of the proof was quite long and difficult and, in a letter to C.A. Borchardt, Hermite declines to enter on a similar research for the number π. This last step was to be passed by Carl Louis Ferdinand von Lindemann (1852-1939), who, in 1882, proved that π is a transcendental number thus ending by a negative answer a question opened for more than 2000 years! The idea of the proof, which uses Padé approximants, is as follows. If $r, s, t, \ldots, z$ are distinct real or complex algebraic numbers and if $a, b, c, \ldots, n$, are real or complex algebraic numbers, at least one of which differing from zero, then $ae^r + be^s + ce^t + \ldots + ne^z \neq 0$. But, $e^{i\pi} + 1 = 0$ and in the preceding result $a = b = 1$ and $c = \ldots = n = 0$; $s = 0$ is algebraic; $r = i\pi$ is the only cause why $e^{i\pi} + 1 = 0$. Since i is algebraic, thus $i\pi$ is transcendental and it follows that π is also transcendental.

In his thesis "*Sur la représentation approchée d' une fonction par des fractions rationelles*", which was presented at the Sorbonne in Paris on June 21, 1892 with the jury: Charles Hermite (Chairman and Advisor), Paul Appell (1855-1930) and Charles Emile Picard (1856-1941), Henri Padé (1863-1953) gave a systematical study of the Padé approximants. He classified them, arranged them in the *Padé table* and investigated the different types of continued fractions whose convergents form a descending staircase or a diagonal in the table. He studied the exponential function in details and showed that its Padé approximants are identical with the rational approximants obtained by Gaston Jean Darboux (1842-1917) in 1876 for the same function. He showed that

$$[n+k/m]_f(t) = \sum_{j=0}^{k-1} c_j t^j + t^k [n/m]_g(t),$$

where $f(t) = c_0 + c_1 t + c_2 t^2 + \ldots$ and $g(t) = c_k + c_{k+1} t + c_{k+2} t^2 + \ldots$, and studied the connection between the two halves of the table. Padé also investigated quite carefully what is now called the block structure of the Padé table.

Using a result given by Jacques Hadamard (1865-1963) in his thesis, Robert Fernand Bernard Viscount de Montessus de Ballore (1870-1937) gave, in 1902, his celebrated result on the convergence of the sequence $\{[n/k]_f : n \in \mathbb{N}_0\}$ where f is a series having k poles and no other singularities in a given circle C. Hadamard's results were extended in 1905 by Paul Dienes (1882-1952). This allowed R. Wilson to investigate in 1927 the behavior of $\{[n/k]_f : n \in \mathbb{N}_0\}$ upon the circle C and at the included poles.

In 1903, Edward Burr Van Vleck (1863-1943) undertook to extend Stieltjes' theory to continued fractions

$$\frac{1}{x+b_1-\frac{a_1}{x+b_2-\frac{a_2}{x+b_3-\ddots}}},$$

where the a_k's are arbitrary positive numbers and the b_k's are arbitrary real numbers. He connected these continued fractions with Stieltjes' type definite integrals with the range of integration taken over the entire real axis. He also extended Stieltjes' theory to Padé table. The name Padé table has been used for the first time by Van Vleck. In 1914, Hilbert's theory of infinite matrices and bounded quadratic forms in infinitely many variables was used by Ernst Hellinger (1883-1950) and Otto Toeplitz (1881-1940) to connect integrals of the form

$$\int_a^b \frac{d\mu(t)}{x-t} \quad (-\infty < a < b < +\infty)$$

with the continued fractions considered by Van Vleck. The same year J. Grommer extended these results to more general cases where the range of integration is the entire real axis. The complete theory was obtained by Hellinger in 1922 using Hilbert's theory of infinite linear systems. The same goal was reached by several other mathematicians (Rolf Hermann Nevanlinna, Torsten Carleman and Marcel Riesz) at about the same time by different methods. Using the results by Van Vleck, Hubert Stanley Wall (1902-1971), in his thesis dated 1927 under Van Vleck's direction, gave a complete analysis of the convergence behavior of the forward diagonal sequences of the Padé table derived from a Stieltjes series, i.e. whose coefficients are given by

$$c_j = \int_0^\infty t^j d\mu(t)\,,$$

with μ bounded and non-decreasing in $[0,+\infty]$. In 1931 and 1932 he extended these results to the cases where the range of integration is $[a,b]$ with $-\infty \le a < b < +\infty$ or with $-\infty < a < b < +\infty$.

The researches on rational approximations during the second part of the twentieth century are mostly devoted to their connection with the theory of orthogonal polynomials and convergence acceleration methods. Since 1965, a growing interest for Padé approximants appeared in theoretical physics, chemistry, electronics, numerical analysis, ... Several international conferences were organized (for example DE BRUIN, M.G. and VAN ROSSUM, H.: *Padé approximation and its applications*. Amsterdam 1980, Lectures Notes in Mathematics 888, Springer Verlag, Heidelberg, 1981; SAFF, E.B. and VARGA, R.S.: *Padé and rational approximation*, Academic Press, New-York, 1977; WUYTACK, L.: *Padé approximation and its applications*, Lectures Notes in Mathematics 765, Springer Verlag, Heidelberg, 1979) and several books were written (for example: BAKER, G.A.jr.: *Essentials of Padé approximants*, Academic Press, New York, 1975; BAKER, G.A. jr. and GRAVES-MORRIS, P.R.: *Padé aprpoximants*, Vols 1 and 2, Encyclopedia of Mathematics and its Applications, Vols. 13 and 14, Addison Wesley, Reading, Mass., 1981; BREZINSKI, C.: *Padé-type approximation and general orthogonal polynomials*, ISNM, Vol. 50, Birskhäuser Verglag, Basel, 1980).

Surely, one the most fundamental and inspired contemporary programs about rational approximation has been that initiated by Claude Brezinski, who was able to extend the notion of Padé approximation by inventing the general theory of Padé-type approximants. Before entering into an explicit outline of his theory in *Section 1.1*, let us understand Brezinski' s motivation. Let f be a formal power series. Padé approximants are rational functions whose expansion in ascending powers of the variable coincides with f as far as possible, that is, up to the sum of the degrees of the numerator and denominator. The numerator and the denominator of a Padé approximant are completely determined by this condition and thus, no freedom is left. If some poles of f are known, it can be interesting to use this information. Padé-type approximants are rational functions with an arbitrary denominator, whose numerator is determined by the condition that the expansion of the Padé-type approximant matches the series f as far as possible, that is, up to the degree of the numerator. It is also possible to choose some of the zeros of the denominator of the Padé-type approximants (instead of all) and then determine the others and the numerator in order to match the series f as far as possible. Such approximants, intermediate between Padé and Padé-type approximants, have been called higher order Padé-type approximants. Padé approximants are a particular case of Padé-type approximants. The great advantage of Padé-type approximants over Padé approximants lies in the free choice of the poles which may lead to a better approximation.

The main open question on Padé-type approximants is the "best" choice of the poles. Some attempts to solve this difficult problem for some particular cases have been made by Alphonse Magnus. Recently, the problem is mostly solved in [49]. Another question connected with the choice of the poles is the convergence of Padé-type approximants. A sufficient answer to this question was given by Michael Eiermann. Several extensions of Brezinski' s ideas are of interest and they are proposed by J. Van Iseghem, A. Draux and M. Prévost.

One would like to adapt the simple proofs of one variable to the several variables case, however major obstacles present themselves. First, the local representation of a function analytic into a domain in $\mathbf{C}^n$ by its Taylor series may lead to extremely complicated and difficult computations. Second, the polydisk does not qualify to be the general target domain because of the failure of the property to be the maximal domain of convergence of a multiple power series. Finally, there is no division process in $\mathrm{P}(\mathbf{C}^n)$, when $n > 1$.

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It is reasonable to suspect that the outlet lies with the consideration of another type of series representation for functions. So, the first *Chapter* of the present book deals with Padé-type (and Padé) approximation to the Fourier series expansion of a harmonic function on the open unit disk D and to the Fourier series expansion of a $2\pi-$periodic L^p-function on $[-\pi,\pi]$ or on the unit circle C.

To give an introductory and brief sketch for the central idea of the *Chapter*, suppose, for instance, $f(t)$ is a $2\pi-$periodic real-valued L^p-function in $[-\pi,\pi]$, with a sequence of Fourier coefficients

$$\{c_\nu : \nu = 0,\pm1,\pm2,...\}.$$

It is clear that $f(t)$can be identified with a real-valued function $u(z)$ in L^p of the unit circle C, by setting $u(e^{it}) = f(t)$. Define the Poisson integral of $u(e^{it})$ by

$$u_r(t) = u(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})P_r(t-\theta)d\theta \quad (0 \le r < 1, -\pi \le t \le \pi),$$

where $\{P_r(\cdot)\}$ is the Poisson kernel in the unit disk. From the solution of the Dirichlet problem in the unit disk D, it follows that the extended real-valued function $u(z) = u(re^{it})$ is harmonic in the open unit disk and such that

$$\lim_{r\to1}\left\|u_r(t) - u(e^{it})\right\|_p = 0.$$

Further, the Fourier series expansion of the restriction $u_r(t)$ of $u(re^{it})$ to any circle C_r of radius $r<1$ is given by

$$\sum_{\nu=-\infty}^{\infty} c_\nu r^{|\nu|} e^{i\nu t}.$$

As we shall show in *Section 1.2*, the Padé-type approximants $\operatorname{Re}(m/m+1)_u(z)$ to the harmonic function $u(z)$ exist and are harmonic real-valued functions in D, such that if the Fourier series expansion of the restriction $\operatorname{Re}(m/m+1)_{u_r}(t)$ of such a Padé-type approximant $\operatorname{Re}(m/m+1)_u(re^{it})$ to the circle of radius $r<1$ is

$$\sum_{\nu=-\infty}^{\infty} d_\nu^{(m)} r^{|\nu|} e^{i\nu t},$$

then, for any $\nu = 0, \pm 1, \pm 2, \ldots, \pm m$, it holds $d_\nu^{(m)} = c_\nu$. Since the radial limit

$$\lim_{r\to 1} \operatorname{Re}(m/m+1)_{u_r}(t)$$

is uniform on $[-\pi, \pi]$, the function

$$\operatorname{Re}(m/m+1)_u(z)$$

is the Poisson integral of a continuous function on the unit circle. This function

$$\operatorname{Re}(m/m+1)_u(e^{it}) = \lim_{r\to 1} \operatorname{Re}(m/m+1)_{u_r}(t) \quad (-\pi \le t \le \pi)$$

is a Padé-type approximant to $f(t)$, in the sense that its Fourier series representation

$$\sum_{\nu=-\infty}^{\infty} d_\nu^{(m)} e^{i\nu t}$$

matches the Fourier series expansion of $f(t)$ up to the $\pm m^{th}-$order's Fourier term. One can also define the Padé approximants $\mathrm{Re}[m/m+1]_f(t)$ to $f(t)$, with Fourier series representation of $\mathrm{Re}[m/m+1]_f(t)$ matching the Fourier series expansion of $f(t)$ up to the $\pm(2m+1)^{th}-$order's Fourier term.

The theoretical study and efficiency of all these approximants constitutes the main purpose of *Section 1.3*. *Paragraph 1.3.1* deals with preparatory material about Dirichlet's problem. The detailed definition and properties for Padé-type and Padé approximants to $2\pi-$periodic real- or complex-valued L^p-functions on the interval $[-\pi,\pi]$ or on the unit circle C are presented in *Paragraph 1.3.2*. *Paragraph 1.3.3* investigates the convergence behavior of a sequence of Padé-type approximants, as well as their connection with Schur and Szegö's theories.

Previously, *Section 1.1*, recalls basic properties of Padé-type and Padé approximation to analytic functions of D, while *Section 1.2* is devoted to the definition, study and examples of Padé-type and Padé approximants to harmonic functions in D.

In Paragraph 1.4.1 of *Section 1.4*, several numerical examples are considered making use of Padé-type approximants to $2\pi-$periodic L^p-functions in $[-\pi,\pi]$, In *Paragraph 1.4.2*, we study the assumptions under which, for every sequence of functions converging to a real-valued continuous $2\pi-$periodic function on $[-\pi,\pi]$, there is a Padé-type approximation sequence converging point-wise to that function faster than the first sequence. Finally, in *Paragraph 1.4.3*, we propose an approximate direct method for the numerical computation of derivatives and definite integrals.

1.1. Rational Approximation to Analytic Functions

1.1.1 Linearized Rational Interpolation and Padé-type Approximants

Interpolating and approximating an analytic function by polynomials or rational functions with prescribed poles is rather well understood and has been studied in great detail by Walsh in [144]. Interpolation by rational functions with pre-assigned poles leads to a theory very similar to that of polynomial interpolation. A rather different situation arises if one considers interpolation by rational functions with free poles. The theoretical background of rational interpolation with free poles is very similar to that of Padé approximants. Actually, Padé approximants are a special type of rational interpolators with all its interpolation points identical.

By f we denote the function which will be interpolated. In the sequel, it is assumed that this function is analytic into the open unit disk D. By $\mathbb{P}_n(\mathbb{C})$ and $\mathbb{R}_{m,n}(\mathbb{C})$ we denote the sets of all complex polynomials of degree at most n and the set of rational functions of numerator and denominator degree at most m and n, respectively.

Let an infinite triangular matrix of interpolation points $\pi_{m,k} \in D$ (called *interpolation scheme*) be given:

$$\mathrm{M} = \begin{pmatrix} \pi_{0,0} & & & \\ \pi_{1,0} & \pi_{1,1} & & \\ \pi_{2,0} & \pi_{2,1} & \pi_{2,2} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Each row

$$\mathrm{M}_m = \{\pi_{m,0}, \pi_{m,1}, \dots, \pi_{m,m}\}$$

of the matrix M defines an interpolation set with $m+1$ interpolation points. It is not excluded that some or all points are identical. With each interpolation set M_m a polynomial

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} (x - \pi_{m,k}) \quad (\gamma \in \mathbb{C} - \{0\})$$

is associated.

Definition 1.1.1.(a). *A rational function $r_{m,n} \in \mathbb{R}_{m,n}(\mathbb{C})$ is called rational interpolator of type (m,n) to the function f at the $m+n+1$ interpolation points of the set M_{m+n}, if*

$$f - r_{m,n} = \mathrm{O}(V_{m+n}) \ \ \textit{at each} \ \ \pi_{m+n,k} \in \mathrm{M}_{m+n} \, .$$

(b). *A rational function*

$$r_{m,n} = \frac{p_m}{q_n} \in \mathbb{R}_{m,n}(\mathbb{C}), \textit{ with } p_m \in \mathbb{P}_m(\mathbb{C}), \ q_n \in \mathbb{P}_n(\mathbb{C}) \textit{ and } q_n \neq 0,$$

is called a linearized rational interpolator (or multi-point Padé approximant) of type (m,n) to the function f at the $m+n+1$ interpolation points of the set M_{m+n}, if

$$q_n f - p_m = \mathrm{O}(V_{m+n}) \ \ \textit{at each} \ \ \pi_{m+n,k} \in \mathrm{M}_{m+n} \, .$$

Definition 1.1.1.(a) implies that at each zero of the polynomial $V_{m+n}(x)$ the interpolation error $f - r_{m,n}$ has a zero of at least the same order. Thus, the interpolator $r_{m,n}$ and its derivatives $(r_{m,n})^{(j)}$ coincide with the function f and its derivatives $f^{(j)}$ at the point $z \in \mathrm{M}_{m+n}$ up to an order determined by the frequency of z in M_{m+n}. *Definition 1.1.1.(a)* therefore defines interpolation in the Hermite's sense. Further, the linearized version in *Definition 1.1.1.(b)* of a rational interpolator $r_{m,n}$ always exists. Indeed, relation $q_n f - p_m = \mathrm{O}(V_{m+n})$ is equivalent to a system of $m+n+1$ linear homogenous equations for the $m+n+2$ unknown parameters (coefficients) in the two polynomials p_m and q_n. Hence, a non-trivial solution always exists, and it is not difficult to verify that for such a solution $q_n \equiv 0$ is impossible.

However, as the next example will show, the existence of a rational interpolator is in general not guaranteed. If, for $m \in \mathbb{N}$, $n \in \mathbb{N}$, a rational function $r_{m,n}$ exists that interpolates f, then one says that the *Cauchy interpolation problem* is solvable ([104]. It is easily verified by comparing two potential candidates that if the interpolation problem is solvable, then the solution is unique.

Example. We choose $m = n = 1$, $\mathrm{M}_2 = \{-1,0,1\}$, and as function to be interpolated $f(z) = z^2$. Any rational function $r_{1,1} \in \mathbb{R}_{1,1}(\mathbb{C})$ is either a Möebius transform or a constant. If $r_{1,1}$ is a Möebius transform, then it is univalent in $\overline{\mathbb{C}}$ and therefore cannot interpolate the value 1 at the two different points -1 and 1. If $r_{1,1}$ is a constant function, then it cannot interpolate the two different values 0 and 1. Hence, already in this very simple situation, a rational interpolator to the function $f(z) = z^2$ does not exist.

Comparison of rational interpolation with interpolation by polynomials shows that the main reason for the non-existence in case of rational interpolators is caused by the non-linearity of the interpolators' parametrization. The first one who mentioned the possibility of non-existence of rational interpolators was Krönecker ([88]). An excellent survey about the solvability of the Cauchy interpolation problem is contained in [104]. There a unified approach to the analysis of the problem is given, which includes elements from the theory of continued fractions, and special matrices and determinants which have been introduced in connection with the interpolation problem, are discussed there. Efficient numerical algorithms that can be applied also in the presence of interpolation defects are discussed in [73].

In what follows, we will give the precise form for rational interpolators of type $(m, m+1)$.

As already mentioned earlier, Padé approximants are a special type of rational interpolators with all its interpolation points identical, that is $\pi_{m+n,0} = \ldots = \pi_{m+n,m+n}$. If, in particular, the $m+m+1$ interpolation points in *Definition 1.1.1* are equal to the origin, that is if $\pi_{m+n,0} = \ldots = \pi_{m+n,m+n} = 0$, then a Padé approximant $[m/n]_f$ to f is a rational fraction whose numerator has the exact degree m and whose denominator has the exact degree n such that its power series agrees with that of f as far as possible:

$$f(z) - [m/n]_f(z) = \mathrm{O}\left(z^{m+n+1}\right) \text{ at } 0.$$

If $[m/n]_f$ exists, then it is unique (apart from a multiplying factor). We shall look for a natural generalization of Padé approximation.

Definition 1.1.3. *A rational function $(m/n)_f(z) \in \mathrm{R}_{m,n}(\mathbb{C})$ is called a Padé-type approximant to the function f if*

$$f(z) - (m/n)_f(z) = \mathrm{O}\left(z^{m+1}\right) \text{ at } 0.$$

There is a good reason for this somewhat strange terminology. According to *Definition 1.1.3*, Padé-type approximants are rational functions with an arbitrary denominator, whose numerator is determined by the condition that the expansion of the Padé-type approximant matches the power series expansion of f as far as possible that is up to the degree of the numerator. On the other hand, Padé approximants are also rational functions whose expansion in ascending powers of the variable coincides with the Taylor expansion of f up to the sum of the degrees of the numerator and denominator. Thus, numerator and denominator of a Padé approximant are completely determined by this condition and no freedom is left. The great advantage of Padé-type approximants over Padé approximants lies in the free choice of the denominator which may lead to a better approximation.

The chief reference on Padé-type approximation is Brezinski' s book [21]. One may also consult [19], [20], [22], [23] and [24].

Let us now see how to construct Padé-type approximants.

We denote the power series expansion of the analytic function f around 0 by

$$f(z) = \sum_{\nu=0}^{\infty} \alpha_{\nu}^{(f)} z^{\nu} \quad (z \in D \Leftrightarrow |z| < 1).$$

If $\mathbb{P}(\mathbb{C})$ is the vector space of all complex analytic polynomials with coefficients in $\mathbb{C}$, we define the linear functional $T_f : \mathbb{P}(\mathbb{C}) \to \mathbb{C}$ associated with f, which satisfies

$$T_f(x^{\nu}) = \alpha_{\nu}^{(f)} \quad (\nu = 0,1,2,\ldots).$$

For the set of all functions complex analytic in an open neighborhood of a given planar set Ω we shall make use of the notation $\mathcal{O}(\Omega)$. The following result is a consequence of Cauchy's integral formula. Two versions of this result in the case of several variables can be found in [38] and [40]. The proof is similar except for the fact that here we also need to consider the Fréchet space $\mathcal{O}(\mathbb{C})$. This consideration is essential and its consequences will be appearing in the sequel.

Theorem 1.1.4.(a). *There is a linear continuous extension of* T_f *on* $\mathcal{O}(\mathbb{C})$.

(b).*There is a linear continuous extension of* T_f *on* $\mathcal{O}(\overline{D})$.

Proof. Let $\{r_n : n = 0,1,2,...\}$ be a sequence of positive numbers such that $r_n < 1$ and let $K_n = \{z \in \mathbb{C}: |z| \le r_n^{-1}\}$ If

$$p(x) = \sum_{\nu=0}^{j} \beta_\nu x^\nu \in \mathbb{P}(\mathbb{C})$$

then there holds

$$|T_f(p(x))| = \left|T_f(\sum_{\nu=0}^{j} \beta_\nu x^\nu)\right| = \left|\sum_{\nu=0}^{j} \beta_\nu T_f(x^\nu)\right| = \left|\sum_{\nu=0}^{j} \beta_\nu \alpha_\nu^{(f)}\right|$$

$$= \left| \sum_{\nu=0}^{j} \frac{\beta_\nu}{2\pi i} \int_{|s|=r_n} \frac{f(s)}{s^{\nu+1}} ds \right| = \left| \frac{1}{2\pi i} \int_{|s|=r_n} \frac{f(s)}{s} \sum_{\nu=0}^{j} \beta_\nu s^{-\nu} \right|$$

$$\le \sup_{|s|=r_n} |f(s)| \sup_{|s|=r_n} |p(s^{-1})| = \sup_{|s|=r_n} |f(s)| \sup_{s\in K_n} |p(s)|,$$

for any n. In the last equality we have used the maximum modulus principle for analytic functions.

(a). If the sequence $\{r_n : n = 0,1,2,...\}$ is strictly decreasing with $\lim_{n\to\infty} r_n = 0$, it is clear that the sets K_n form an exhaustive sequence of compacts in $\mathbb{C}$ and then the *Hahn-Banach Theorem* extends T_f to a linear continuous functional of $\mathcal{O}(\mathbb{C})$ when considered with the usual topology of uniform convergence on the compact subsets of $\mathbb{C}$.

(b). By density, if $\lim_{n\to\infty} r_n = 1$, then, there is a continuous linear extension of T_f on $\mathcal{O}(\overline{D})$.

The first consequence of this *Theorem* is described in the following

Corollary 1.1.5. *For every* $z \in D$, *the number* $T_f\left((1-xz)^{-1}\right)$ *is well defined and equals* $f(z)$.

*Proof***.** Let $z \in D$. By *Theorem 1.1.4*, the number

$$T_f\left((1-xz)^{-1}\right)$$

is well defined (:T_f acts on the variable $x \in \overline{D}$ and $z \in D$ is regarded as a parameter). The continuity of T_f implies

$$f(z)=\sum_{\nu=0}^{\infty} a_\nu^{(f)} z^\nu = \sum_{\nu=0}^{\infty} T_f(x^\nu) z^\nu = \sum_{\nu=0}^{\infty} T_f(x^\nu z^\nu) = T_f(\sum_{\nu=0}^{\infty} x^\nu z^\nu) = T_f\left((1-xz)^{-1}\right).$$

A second consequence of *Theorem 1.1.4* is that the functional T_f has a linear continuous extension into the space $\mathcal{M}(\mathbb{C})$ of meromorphic functions in $\mathbb{C}$. In fact, we have $\mathcal{M}(\mathbb{C})=C(\mathbb{C}, S^2)$, where $C(\mathbb{C}, S^2)$ is the space of continuous mappings from $\mathbb{C}$ into the Riemann sphere S^2. Since S^2 is a metric space with respect to the chordal distance, we can consider in $C(\mathbb{C},S^2)$ the topology of uniform convergence on compact subsets of $\mathbb{C}$. When restricted to the subspace $\mathcal{O}(\mathbb{C})$ of $C(\mathbb{C}, S^2)$, this topology actually coincides with the usual topology of $\mathcal{O}(\mathbb{C})$. More precisely, a sequence $\{f_n : n = 0,1,2,...\} \in C(\mathbb{C}, S^2)$ is said to *converge normally* to a function $F: \mathbb{C} \to S^2$ if $\sigma(f_n, F) \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{C}$, where σ is the chordal distance in S^2. The crucial property is that $C(\mathbb{C},S^2)$ is a Fréchet space when considered with the topology of normal convergence. Moreover, if all the f_n are analytic and $F \neq \infty$, then $F \in \mathcal{O}(\mathbb{C})$ and

$f_n \to F$ in the topology of $\mathcal{O}(\mathbb{C})$. The *Hahn-Banach Theorem* combined now with T*heorem 1.1.4.(a)* shows immediately that

Corollary 1.1.6. *There is a continuous linear extension of* T_f *into* $C\,(\mathbb{C}, \mathrm{S}^2)$.

Next, suppose again

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m\geq 0, 0\leq k\leq m}$$

is an infinite triangular interpolation matrix with $\pi_{m,k} \in D$. With each row

$$\mathrm{M}_m = \left\{\pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}\right\}$$

a polynomial

$$V_{m+1}(x) = \gamma\sum_{k=0}^{m}(x-\pi_{m,k}) \quad (\gamma \in \mathbb{C}-\{0\})$$

is associated.

Let $m \geq 0$ be fixed. For any fixed $z \in \mathbb{C}-\{\pi_{m,k}^{-1} : k = 0,1,\ldots,m\}$, let $Q_m(x,z)$ denote the unique polynomial of degree at most m which interpolates $(1-xz)^{-1}$ in the $m+1$ nodes of the m^{th} row M_m of M, i.e.

$$Q_m(\pi_{m,k}, z) = (1-\pi_{m,k}z)^{-1} \quad (k = 0,1,2,\ldots,m).$$

If some of the nodes $\pi_{m,k}$ coincide the interpolation has to be understood in the Hermite sense. Obviously, the expression

$$W_m(z) = T_f\left(\frac{V_{m+1}(x) - V_{m+1}(z)}{x - z}\right)$$

is a polynomial in z of degree at most m (here the functional T_f acts on the variable x and z is taken as a parameter. By using the partial fraction decomposition of

$$z^{W_m(z)}\!\Big/_{V_{m+1}}(z),$$

one obtains

$$T_f(Q_m(x,z)) = \frac{W_m^*(z)}{V_{m+1}^*(z)},$$

with

$$W_m^*(z) := z^m W_m(z^{-1}) \text{ and } V_{m+1}^* := z^{m+1} V_{m+1}(z^{-1}).$$

Since $W_m^*(z)$ and $V_{m+1}^*(z)$ are two polynomials in z with degrees at most m and $m+1$, respectively, we see that $T_f(Q_m(x,z))$ is a rational function in z of type $(m, m+1)$, which means that it has a numerator with degree at most m and a denominator with degree at most $m+1$. The basic property is that

Theorem 1.1.7.

$$f(z) - T_f(Q_m(x,z)) = \mathrm{O}(z^{m+1}),$$

for any $z \in D$ *, and therefore*

$$(m/m+1)_f(z) = T_f(Q_m(x,z)) = \frac{W_m^*(z)}{V_{m+1}^*(z)}.$$

One can also construct Padé-type approximants with various degrees in numerators. For all $n = 1,2,3,\ldots$, the Taylor power series expansion of $f(z)$ can be rewritten as

$$f(z) = \sum_{\nu=0}^{n-1} \alpha_{\nu}^{(f)} z^{\nu} + z^{n} f_{n}(z) = \sum_{\nu=0}^{n-1} \alpha_{\nu}^{(f)} z^{\nu} + z^{n} \sum_{\nu=0}^{\infty} \alpha_{\nu+k}^{(f)} z^{\nu} \ , \ z \in D \ .$$

Clearly, the rational function

$$\sum_{\nu=0}^{n-1} \alpha_{\nu}^{(f)} z^{\nu} + T_{f_n}\left(Q_m(x,z)\right)$$

is the type $(m+n, m+1)$. Its denominator is again given by $V_{m+1}^{*}(z) = z^{m+1} V_{m+1}(z^{-1})$ and therefore, it has poles in the inverse nodes $\pi_{m,k}^{-1}$ $(k = 0,1,2,\ldots m)$. As before, it can be shown that

Theorem 1.1.8.

$$f(z) - \{\sum_{\nu=0}^{n-1} \alpha_{\nu}^{(f)} z^{\nu} + T_{f_n}\left(Q_m(x,z)\right)\} = \mathrm{O}(z^{m+1}), \ \textit{for} \ z \in D \ ,$$

and therefore

$$(m+n/m+1)_f(z) = \{\sum_{\nu=0}^{n-1} \alpha_{\nu}^{(f)} z^{\nu} + T_{f_n}\left(Q_m(x,z)\right)\}.$$

For later use, we shall say that $(m/m+1)_f$ and $(m+n/m+1)_f$ are two *Padé-type approximants to* f *with generating polynomial* $V_{m+1}(x)$.

The main open question on Padé-type approximants is the "best" choice of the poles that is the "best" choice of the generating polynomials' roots $\pi_{m,k}$. Some attempts to solve this difficult problem for some particular cases have been made by Magnus in [103]. In [49], we have determined the "best" choice of points $\pi^*_{m,0} = \pi^*_{m,0}(z), \pi^*_{m,1} = \pi^*_{m,1}(z), \ldots, \pi^*_{m,m} = \pi^*_{m,m}(z)$ for the interpolation system $\pi_{m,0}, \pi_{m,1}, \ldots \pi_{m,m}$, in the sense that the corresponding Hermite polynomial $Q^*_m(x,z)$ minimizes *point-wise* in *z* the absolute error

$$\left| f(z) - (m/m+1)_f(z) \right|, (\, f \in O(D)\,)$$

into the ring $0 < |z| < 1$,and, on the other hand, we have showed that if m =*even*, the same as above choice constitutes also the "best" L^2-choice, in the sense that it minimizes the L^2-norm

$$\left\| f(z) - (m/m+1)_f(z) \right\|_2^{\delta,\varepsilon} = \left(\int\limits_{\delta \le |z| < \varepsilon} \left| f(z) - (m/m+1)_f(z) \right|^2 dz \right)^{1/2}$$

of the error into an arbitrary half-open ring $\Delta(0;\delta,\varepsilon) := \{ z \in \mathrm{C} : \delta \le |z| < \varepsilon \}$,with $0 < \delta < \varepsilon < 1$, over the subset of all Hermite polynomials $Q_m(x,z)$, that is

$$\left\| f(z) - (m/m+1)_f(z) \right\|_2^{\delta,\varepsilon} = \min_{\pi_{m,0},\pi_{m,1},\ldots,\pi_{m,m}} \left\| f(z) - (m/m+1)_f(z) \right\|_2^{\delta,\varepsilon}, f \in O(D)$$

In both cases, the interpolation polynomial $Q^*_m(x,z)$ has the form:

$$Q^*_m(x,z) = \sum_{\nu=0}^{m-1} z^\nu x^\nu + x^m \quad (z \neq 0, x \in \mathrm{C}),$$

and the interpolation points $\pi^*_{m,0} = \pi^*_{m,0}(z), \ldots, \pi^*_{m,m} = \pi^*_{m,m}(z)$ are the $(m+1)$ roots of the generating polynomial

$$V_{m+1}^{*}(x) \equiv_{(z)} V_{m+1}^{*}(x) := x^{m+1} + \frac{1}{z}(z^{m} - 1)x^{m}.$$

Further, if $m = even$, the Hermite polynomial $Q_m^*(x,z)$ is the unique interpolation polynomial of degree at most m which minimizes the number

$$\left(\int\limits_{\delta \le |z| < \varepsilon} \int\limits_{|s| = \frac{1}{r}} \left| \frac{1}{1-sz} - Q_m(s,z) \right|^2 dsdz \right)^{1/2}$$

and satisfies

$$\int\limits_{\delta \le |z| < \varepsilon} \int\limits_{|s| = \frac{1}{r}} \left(\frac{1}{1-sz} - Q_m^*(s,z) \right) dsdz = 0 \quad (0 < \delta < \varepsilon < r < 1).$$

In spite of these results, there is no analogous possibility to determine a "best" uniform choice for the interpolation system $\pi_{m,0}, \pi_{m,1}, ..., \pi_{m,m}$, since a minimum for the uniform norm

$$\left\| f(z) - (m/m+1)_f(z) \right\|_{\infty}^{\delta,\varepsilon} := \sup_{\delta \le |z| \le \varepsilon} \left| f(z) - (m/m+1)_f(z) \right| \quad (f \in O(D))$$

of the error on a compact ring $\overline{\Delta(0;\delta,\varepsilon)}$ is obtained at the limit points

$$\pi_k = \infty \; (k = 0,1,...,m-i) \quad \text{and} \quad \pi_\nu = 0 \; (\nu = m-i+1,...,m)$$

for any $i = 0,1,2,...,m+1$. In particular, among these choice, the only feasible optimal interpolation system is given by

$$\pi_0 = \pi_1 = ... = \pi_m = 0.$$

Let us now turn to the investigation of a precise form for rational interpolators of type $(m/m+1)_f$.

Theorem 1.1.9. *Suppose*

$$\pi_{2m+1,m+1} = \pi_{2m+1,m+2} = \ldots = \pi_{2m+1,2m+1} = 0.$$

Set

$$u_{m+1}(x) := \prod_{k=0}^{m} (x - \pi_{2m+1,k}) \quad \text{and} \quad u_{m+1}^{*}(z) := z^{m+1} u_{m+1}(z^{-1}) \ .$$

(a). *If*

$$r_{m,m+1} = \frac{q_m}{u_{m+1}^{*}} \in \mathrm{R}_{m,m+1}(\mathbb{C})$$

is a linearized rational interpolator to the analytic function f at the 2m+2 interpolation points of the set M_{2m+1}, *then* $r_{m,m+1}$ *is a Padé-type approximant* $(m/m+1)_f$ *to* f *with generating polynomial* $u_{m+1}(x)$.

(b). *Conversely, a Padé-type approximant* $(m/m+1)_f$ *to* f *with generating polynomial* $u_{m+1}(x)$ *is a linearized rational interpolator* $r_{m,m+1} \in \mathrm{R}_{m,m+1}(\mathbb{C})$ *to* f *at the* $2m+2$ *interpolation points of the set* M_{2m+1}, *if*

$$\int_{|s|=r} \frac{f(s)}{s^{2m+3}} \prod_{k=0(k\neq j)}^{m} (1 - s\pi_{m,k}) ds = 0 \qquad (j = 0,1,2,\ldots,m)$$

for some $r < 1$.

Proof. (a). Since each of the functions

$$\frac{z-\pi_{2m+1,k}}{1-z\pi_{2m+1,k}}$$

is of modulus 1 on the unit circle, we see that

$$\left|\frac{z-\pi_{2m+1,k}}{1-z\pi_{2m+1,k}}\right| \le 1 \Leftrightarrow \left|z(z^{-1}-\pi_{2m+1,k})\right| \ge \left|z-\pi_{2m+1,k}\right|$$

for any $z \in D$ and $k = 0,1,2,...,m$. It follows that

$$\left|z^{2m+2}\prod_{k=0}^{m}\left(z^{-1}-\pi_{2m+1,k}\right)\right| \ge \left|\prod_{k=0}^{2m+1}\left(z-\pi_{2m+1,k}\right)\right|.$$

If $r_{m,m+1} = \dfrac{q_m}{u^*_{m+1}} \in \mathcal{R}_{m,m+1}(\mathbb{C})$ is a linearized rational interpolator to f at the $2m+2$ points of M_{2m+1}, then

$$\left|\frac{u^*_{m+1}(z)f(z)-q_m(z)}{\prod_{k=0}^{2m+1}\left(z-\pi_{2m+1,k}\right)}\right| \le C \Rightarrow \left|\frac{u^*_{m+1}(z)f(z)-q_m(z)}{z^{2m+2}\prod_{k=0}^{m}\left(z^{-1}-\pi_{2m+1,k}\right)}\right| \le C \Rightarrow$$
$$\left|\frac{u^*_{m+1}(z)f(z)-q_m(z)}{z^{m+1}u^*_{m+1}(z)}\right| \le C \Rightarrow \left|f(z)-r_{m,m+1}(z)\right| \le C\left|z^{m+1}\right|,$$

for any $z \in D$ and some constant C. By *Definition 1.1.3*, the rational function $r_{m,n+1}$ is a Padé-type approximant to f with generating polynomial $u_{m+1}(x)$.

(b). The error of the Padé-type approximation to f with generating polynomial

$$u_{m+1}(x)=\sum_{j=0}^{m}\beta_j x^j$$

is given by

$$f(z)-(m/m+1)_f(z)=\frac{z^{2m+2}}{u_{m+1}^*(z)}T_f\left(\frac{x^{m+1}u_{m+1}(x)}{1-xz}\right)$$

$$=\frac{z^{2m+2}}{u_{m+1}^*(z)}T_f\left(\sum_{\nu=0}^{\infty}x^{m+\nu+1}u_{m+1}(x)z^\nu\right)$$

$$=\frac{z^{2m+2}}{u_{m+1}^*(z)}T_f\left(\sum_{\nu=0}^{\infty}x^{m+\nu+1}\sum_{j=0}^{m}\beta_j x^j z^\nu\right)$$

$$=\frac{z^{2m+2}}{u_{m+1}^*(z)}\sum_{\nu=0}^{\infty}\left[\sum_{j=0}^{m}\beta_j\alpha_{m+j+\nu+1}^{(f)}\right]z^\nu$$

$$=\frac{z^{2m+2}}{2\pi i u_{m+1}^*(z)}\sum_{\nu=0}^{\infty}[\ [\int_{|s|=r}\left(\sum_{j=0}^{m}\beta_j s^{-k}\right)\frac{f(s)}{s^{m+\nu+2}}ds\ ]z^\nu$$

$$=\frac{z^{2m+2}}{2\pi i u_{m+1}^*(z)}\sum_{\nu=0}^{\infty}[\ \int_{|s|=r}u_{m+1}\left(s^{-1}\right)\frac{f(s)}{s^{m+\nu+2}}ds\ ]z^\nu$$

$$=\frac{z^{2m+2}}{2\pi i u_{m+1}^*(z)}\int_{|s|=r}\frac{f(s)}{s^{m+2}}u_{m+1}(s^{-1})\sum_{\nu=0}^{\infty}s^\nu z^\nu ds$$

$$=\frac{z^{2m+2}}{2\pi i u_{m+1}^*(z)}\int_{|s|=r}\frac{f(s)u_{m+1}\left(s^{-1}\right)}{s^{m+2}(1-sz)}ds,\qquad (z\in D)$$

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for some $r < 1$. Obviously, for any $k = m+1, m+2,...,2m+1$, there holds

$$f\left(\pi_{2m+1,k}\right) = \left(m/m+1\right)_f\left(\pi_{2m+1,k}\right),$$

and $\left(m/m+1\right)_f$ interpolates f at $\pi_{2m+1,0}, \pi_{2m+1,1},...,\pi_{2m+1,m}$, if

$$\int_{|s|=r} \frac{f(s)\prod_{k=0}^{m}\left(s^{-1}-\pi_{2m+1,k}\right)}{s^{m+2}\left(1-s\pi_{2m+1,j}\right)} \quad \text{for any } j = 0,1,...,m$$

$$\Leftrightarrow \qquad \int_{|s|=r} \frac{f(s)\prod_{k=0}^{m}\left(s^{-1}-\pi_{2m+1,k}\right)}{s^{m+2}(1-s\pi_{2m+1,j})} ds = 0 \qquad \text{for any } j = 0,1,...,m$$

$$\Leftrightarrow \qquad \int_{|s|=r} \frac{f(s)}{s^{2m+3}} \prod_{k=0(k\neq j)}^{m}\left(1-s\pi_{2m+1,k}\right)ds = 0 \qquad \text{for any } j = 0,1,...,m,$$

which completes the *Proof* of the *Theorem*.

A more general result is still missing. However, because of the above *Theorem*, in the present work we are not really concerned with properties of rational interpolators; our primary interest in this *Section* will be the definition of Padé and Padé-type approximants to harmonic functions in the unit disk D and to L^p − functions on the unit circle C (or on the interval $\left[-\pi,\pi\right]$). The convergence behavior of these approximants will also be of our interest, and, in this direction, the ingredient key will be the understanding of the solution to the corresponding problem in the analytic function case.

1.1.2. The Convergence Problem

If the numerator and denominator's degrees of the Padé-type approximants grow, then the questions arise whether and where the approximants converge to the function that has been approximated. Orientation for answers can be obtained from convergence results proved for Padé approximants. There compact (i.e., locally uniform) convergence has been proved for certain classes of functions. In the present *Paragraph*, we first discuss corresponding results for rational interpolation. In comparison to later topics, we shall do this in a rather compressed and summarizing form. The discussion is followed by a study of the convergence problem of Padé-type approximants. In all cases our interest is restricted to diagonal or close-to-diagonal sequences of interpolators and approximants, i.e., interpolators and approximants with numerator degree m equal or almost equal to the denominator's degree.

In the convergence theory of Padé-type approximants functions of the form

$$F_\mu(z) = \int \frac{d\mu(x)}{x-z}$$

with μ a positive measure supported on $\mathbb{R}$ play a prominent role. They are known as Markov, Stieltjes, or Hamburger functions, depending on whether $\sup p(\mu)$ is compact, contained in one of the two half-axis $(-\infty,0]$ or $[0,+\infty)$, or unbounded and intersecting with both sets $(-\infty,0]$ and $[0,+\infty)$, respectively. Diagonal Padé approximants developed at infinity to functions $F_\mu(z)$ converge compactly (,i.e., locally uniformly) in the domain $\overline{\mathbb{C}} - \overline{\sup p(\mu)}$. In case of Stieltjes or Hamburger functions it is necessary in addition that the moment problem associated with the measure μ is determinate ([4]).

Analogous results for rational interpolators to functions $F_\mu(z)$ have been proved in [66], [67], [98], [99], [100], [101] and [135].

If q_m is the denominator polynomial of the Padé approximant $[m/m]_{F_m}$ to a function F_μ developed at infinity, then the polynomial q_m is orthogonal with respect to the measure μ ([135]). The denominator polynomial q_m is characterized by this orthogonality property up to a constant factor. A similar characterization of the denominator holds in case of a rational interpolator $r_{m,m}(z)$ to $F_\mu(z)$, however, now the denominator polynomial is orthogonal with respect to a weighted orthogonality relation:

$$\int x^\nu q_m(x) \frac{d\mu(x)}{V_{2m}(x)} = 0 \quad \text{for } \nu = 0,1,2,\dots,m-1,$$

where $V_{2m}(x)$ is the polynomial

$$\prod_{k=0}^{2m-1} \left(x - \pi_{2m-1,k}\right) \text{ ([135]).}$$

Thus, q_m is orthogonal with respect to the weighted measure $V_{2m}^{-1} d\mu$. In nearly all respects the convergence theory of rational interpolators to functions F_μ is a direct generalization of that of Padé approximants. In both cases, the convergence domain is $\bar{\mathbb{C}} - \overline{\sup p(\mu)}$, and for the interpolation error the asymptotic estimate

$$\limsup_{m\to\infty} \left|F_\mu(z) - r_{m,m}(z)\right|^{1/2m} \le \exp\left(-\int g_{\bar{\mathbb{C}} - \overline{\sup p(\mu)}}(z,x)\, d\alpha_{\mathrm{M}}(x)\right) \quad \text{([135])}$$

holds for $z \in \bar{\mathbb{C}} - \overline{\operatorname{supp}(\mu)}$, where $g_{\bar{\mathbb{C}} - \overline{\sup p(\mu)}}(z,x)$ is the Green function of the domain $\bar{\mathbb{C}} - \overline{\operatorname{supp}(\mu)}$ and where $d\alpha_{\mathrm{M}}(\cdot)$ is the asymptotic distribution of the matrix M, that is the probability measure which satisfies

$$\lim_{m\to\infty}\left(\frac{1}{m+1}\sum_{k=0}^{m}\delta_{\pi_{m,k}}\right)=\alpha_{\mathrm{M}}$$

with respect to the weak topology in the space of Borel measures ($\delta_{\pi_{m,k}}$ is the Dirac measure at $\pi_{m,k}$). Under certain conditions the above estimate is sharp.

Another class of functions, for which compact convergence of Padé approximants has been proved, are the Pölya frequency functions

$$G(z)=e^{\gamma z}\frac{\prod_j(1+\alpha_j z)}{\prod_j(1+\beta_j z)}$$

where $\gamma,\alpha_j,\beta_j \geq 0$ and

$$\sum_j(\alpha_j+\beta_j)<\infty .$$

It has been shown in [3] that diagonal Padé approximants to these functions converge compactly in $\overline{\mathbb{C}}-\{-\alpha_1^{-1}.,-\alpha_2^{-1},....,\beta_1,\beta_2,...\}$ In [6] this result has been extended to rational interpolators with interpolation matrices that contain only real entries $\pi_{m,k}$ and the functions G can have only finitely many factors in their definition. The general problem is still open.

From counterexamples involving Padé approximants, we know that the analyticity of the function f is not sufficient for guaranteeing compact convergence of rational interpolators. In [142] it has been shown that it is possible to construct an entire function such that the diagonal sequence of its Padé approximants developed at the origin diverges at each point of $\overline{\mathbb{C}}-\{0\}$. Thus, this counterexample underlines that in the convergence results for the classes of functions F_j and G the special structure of these functions is crucial.

Having the difficulties with compact convergence in mind, it is certainly interesting to realize that convergence can be proved for large classes of functions, which are defined mainly by analyticity properties, if a weaker type of convergence is considered. Especially successful has proved convergence in capacity.

By $cap(\cdot)$ we denote the (logarithmic) capacity of (capacitable) subsets of $\mathbb{C}$ (for a definition see [90], [135] and [139]). For any Borel set $B \subseteq \mathbb{C}$, we have $d\lambda(B) \leq \pi\, cap^2(B)$, where $d\lambda(\cdot)$ denotes the planar Lebesgue measure. This inequality shows that sets that are small in capacity are also small in planar Lebesgue measure. A sequence of functions f_m $(m = 0,1,2,...)$ is said to *converge in capacity* to f in the disk D, if for every $\varepsilon > 0$ and every compact set $E \subset D$ we have

$$\lim_{m\to\infty} cap\left(\{z \in E : \left|f_m(z) - f(z)\right| > \varepsilon\}\right) = 0.$$

The first result about convergence in capacity and Padé approximation was proved in [120] after preparations in [112]. In [143] the *Nuttall-Pommerenke Theorem* has been extended to rational interpolators:

Theorem 1.1.10. *Let the function f be analytic (and single-valued) in the domain $D - K$ with K a compact set of $cap(K) = 0$ and such that*

$$K \cap \overline{\{z \in \mathbb{C} : z \in \mathrm{M}_m, m \in \mathbb{N}\}} = \varnothing,$$

and let $r_{m,m}$ be the linearized rational interpolator to the function f in the points of the set M_{2m}. Then, for every compact set $E \subset D$ and every $\varepsilon > 0$, we have

$$\lim_{m\to\infty} cap\{z \in E : \left|r_{m,m}(z) - f(z)\right| > \varepsilon^m\} = 0.$$

It follows that the sequence of rational interpolators $r_{m,m}$ $(m = 1,2,...)$ converges in capacity to f in D. (But even more, we see that the convergence speed is faster than geometric with possible exceptions on sets that become small in capacity as $m \to \infty$.)

In [143], the *Nuttall-Pommerenke Theorem* has been proved not only for the diagonal sequence $\{r_{m,m} : m \in \mathbb{N}\}$, but also for arbitrary sectorial sequences, i.e., for sequences $\{r_{m,n} : m \in \mathbb{N},\ n \in \mathbb{N}\}$ with a $\lambda > 0$ such that

$$\lambda \leq \frac{m}{n} \leq \frac{1}{\lambda} \text{ as } m,n \to \infty..$$

The assumption $cap(K) = 0$ is essential for the *Proof* of *Theorem 1.1.10*. In [102] and [122] it has been shown by counterexamples that if the function f has a set of singularity of positive capacity, then convergence in capacity can no longer be guaranteed for diagonal Padé approximants in any sub-domain of D.

All meromorphic functions f satisfy the assumptions of *Theorem 1.1.10*, but the functions covered by the theorem form a much larger class. For instance, the functions f may have essential singularities as long as there are not too many of them. Of course, any entire function is covered by *Theorem 1.1.10*.

In [105] it has been shown that convergence in capacity implies point-wise convergence quasi everywhere for appropriately chosen infinite subsequences. In analogy to the notion “almost everywhere”, a property is said to hold “*quasi everywhere*” on a set S if it holds for every $z \in S$ with possible exceptions on sets of outer capacity zero.

Especially for rational interpolators which are Padé-type approximants to a function $f \in \mathcal{O}(D)$, there are some general and sufficient conditions determining the compact convergence in D.

The first result in this direction can be viewed as a consequence of results due to Wimp [145] by using Newton's relations between the coefficients and the zeroes of a polynomial:

Proposition 1.1.11. *Let* $\{\pi_k : k = 0,1,2,...\}$ *be a sequence of numbers in* D *and let*

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} (x - \pi_k)$$

be the generating polynomial of the Padé-type approximant $(m+n/m+1)_f(z)$ *to a function* $f \in \mathcal{O}(D)$.

A sufficient condition to hold

$$\lim_{m\to\infty}(m+n/m+1)_f(z) = f(z) \quad in \ D \quad (n = 1,2,...)$$

is the convergence of the series

$$\sum_{k=0}^{\infty} \frac{z\pi_k}{1 - z\pi_k} \quad in \ D.$$

Let us give a second particular result:

Proposition 1.1.12. ([145]) *Let* $\{\pi_k : k = 0,1,2,...\}$ *be a sequence of negative numbers in* D *converging to* 0 *and let*

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} (x - \pi_k)$$

be the generating polynomial of the Padé-type approximant $(m+n/m+1)_f(z)$ *to a function* $f \in \mathcal{O}(D)$.

For any $z \in D \cap [0,1]$, *there holds*

$$\lim_{m\to\infty}(m+n/m+1)_f(z) = f(z) \qquad (n = 1,2,...).$$

The application of summability methods is a classical tool of analytic continuation and in this connection we are going to prove a very general result:

Theorem 1.1.13. *If the generating polynomials*

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} \left(x - \pi_{m,k}\right)$$

satisfy

$$\lim_{m\to\infty} \frac{V_{m+1}(x)}{V_{m+1}(z^{-1})} = 0$$

compactly in an open set $\omega \subset \mathbb{C}^2$ *containing* $\mathbb{C} \times \{0\}$, *then*

$$\lim_{m\to\infty} (m/m+1)_f(z) = f(z)$$

compactly in $\{z \in D : (\zeta, z) \in \omega,$ *for any* $|\zeta| \le 1\}$.

This result was first proved by Eiermann in [55] and in case of several complex variables in [38].

Suppose we start with a sequence

$$\mathrm{N}(z) = \left(\sigma_{m,k}(z)\right)_{m\ge 0, 0\le k\le m}$$

of complex-valued functions in D, and let

$$\{t_m^{(f)}(z) = \sum_{k=0}^{m} \sigma_{m,k}(z) \sum_{\nu=0}^{k} \alpha_\nu^{(f)} z^\nu : z \in D \text{ and } m = 0,1,2,\ldots\}$$

be the $\mathrm{N}(z)-$transform of the sequence of the partial sums of f around 0. Consider the sequence of functions of two complex variables:

$$d_m(x,z) = \sum_{k=0}^{m} \sigma_{m,k}(z) \sum_{\nu=0}^{k} x^\nu z^\nu : (x,z) \in \mathbb{C}^2 \text{ and } m = 0,1,2,\ldots\},$$

and define $\omega(\mathrm{N})$ to be an open subset of $\mathbb{C}^2$ into which this sequence converges compactly to $(1 - xz)^{-1}$.

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With these definitions we can state the following:

Theorem 1.1.14. *If* $\omega(\mathrm{N}) \subset \mathbb{C} \times \{0\}$, *then for any* $f \in \mathcal{O}(D)$ *the* $\mathrm{N}(z)-$*transform of the sequence of the partial sums of* f *around* 0 *converges to* $f(z)$ *compactly on*

$$g(\omega(\mathrm{N}(z)) := \{z \in D : (\zeta, z) \in \omega(\mathrm{N}), \textit{for any } |\zeta| \leq 1\}.$$

This *Theorem* constitutes a generalized form of *Okada Theorem* (see [55], [62], [37], [38], [19] and [51].

As a consequence, we can immediately prove *Theorem 1.1.13*. Indeed, if we choose the sequence $\mathrm{N}(z)$ in such a way to have $d_m(x,z) = Q_m(x,z)$ for any m, then

$$t_m^f(z) = T_f(d_m(x,z)) = T_f(Q_m(x,z)) = (m/m+1)_f(z)$$

and it is enough to show that the open set $\omega \subset \mathbb{C}^2$, into which the sequence $\{V_{m+1}(x)V_{m+1}^{-1}(z) : m = 0,1,2,...\}$ converges compactly to 0, is contained in $\omega(\mathrm{N}) \subset \mathbb{C}^2$, the set into which the sequence $\{Q_m(x,z) : m = 01,2,...\}$ converges compactly to $(1-xz)^{-1}$. Since

$$\frac{1}{1-xz} - Q_m(x,z) = \frac{1}{1-xz}\frac{V_{m+1}(x)}{V_{m+1}(z^{-1})},$$

the assumption that

$$\lim_{m\to\infty} \frac{V_{m+1}(x)}{V_{m+1}(z^{-1})} = 0$$

compactly guarantees that

$$\lim_{m\to\infty} Q_m(x,z) = (1-xz)^{-1}$$

(and conversely). This proves *Theorem 1.1.13*.

The *Proof* of *Theorem 1.1.14* requires the following *Lemma.*

Lemma 1.1.15. *Let* D_r *be open disk centered at* 0 *and with radius* r *and let* $K \subset\subset D_r \subset\subset D$. *Suppose the functions* $d_m(x,z)$ *are continuous in the set*

$$[bD_r]^{-1} \times K := \{(x^{-1}, z) : |x| = r, z \in K\}.$$

If $f \in \mathcal{O}(D)$, *then for all* m *there holds*

$$\sup_{z\in K}\left|f(z) - T_f(d_m(x,z))\right| \leq \mathcal{L}_f \sup_{(t,z)\in\{bD_r\}^{-1}\times K}\left|(1-tz)^{-1} - d_m(t,z)\right|,$$

where the constant $\mathcal{L}_f$ *depends only on* f *and* r, *but is independent of* m *and* K.

Assuming the *Lemma* for a moment, let us prove *Theorem 1.1.14*.

Proof of Theorem 1.1.14. Let $f \in \mathcal{O}(D)$. It is sufficient to show that for every $z^\circ \in g(\omega(\mathrm{N}))$ there exists a closed disk $\overline{\Delta(z^\circ, p^\circ)}$ centered at z° and with radius p° such that

$$\overline{\Delta(z^\circ, p^\circ)} \subset g(\omega(\mathrm{N}))$$

and

$$\lim_{m\to\infty} T_f(d_m(x,z)) = f(z) \quad \text{uniformly on} \quad \overline{\Delta(z^\circ, p^\circ)} \subset g(\omega(\mathrm{N})).$$

We must distinguish two cases:

$$z^\circ \neq 0 \quad \text{and} \quad z^\circ = 0.$$

Let $z^{\circ} \neq 0$. Since $g(\omega(N))$ is open, one can find a $p^{\circ} > 0$ such that $\overline{\Delta(z^{\circ}, p^{\circ})} \subset g(\omega(\mathrm{N}))$. By the definition of $g(\omega(N))$, the compact set

$$\{(x,z) \in \mathbb{C}^2 : |x| \leq 1, z \in \overline{\Delta(z^{\circ}, p^{\circ})}\}$$

is contained in the open set $\omega(\mathrm{N})$. It follows that there is a $\varepsilon > 0$ with

$$\{(x,z) \in \mathbb{C}^2 : |x| \leq 1, z \in \overline{\Delta(z^{\circ}, p^{\circ})}\} \subset \omega(\mathrm{N}).$$

Since $\overline{\Delta(z^{\circ}, p^{\circ})} \subset g(\omega(\mathrm{N})) \subset D$, there holds

$$\varepsilon' := dist\left[\{\xi \in D : |\xi| = \max[1-\varepsilon, \sup_{|z-z^{\circ}| \leq p^{\circ}} |z|]\}, bD\right] > 0.$$

By defining $r = 1 - \varepsilon'$, we see that the *Lemma 1.1.15* can be applied to the disk D_r and the compact set $K = \overline{\Delta(z^{\circ}, p^{\circ})}$; and thus the desired conclusion follows.

Let now $z^{\circ} = 0$. Choose $r > 0$ so that the closed disk $\overline{D_r}$, with center 0 and radius r, is contained in D. By assumption, the compact set

$$\{(x,0) \in \mathbb{C}^2 : |x| \leq r\}$$

is contained in the open set $\omega(\mathrm{N})$. Hence,

$$\{(x,z) \in \mathbb{C}^2 : |x| \leq r, |z| \leq \tau\}$$

for a suitably small chosen $\tau > 0$. It is obvious that the *Lemma 1.1.15* can be applied to the open disk D_r and the compact set $K = \overline{\Delta(0, p^{\circ})}$, where

$$0 < p^{\circ} < \min\{\tau, r\}$$

and the *Proof* of *Theorem 1.1.14* is complete.

Proof of Lemma 1.1.15. The *Proof* is a direct consequence of *Cauchy's Integral Formula*. If $f \in \mathcal{O}(D)$, then, by *Corollary 1.1.5*, we have

$$\left|f(z)-T_f(d_m(x,z))\right| = \left|T_f\left((1-xz)^{-1}-d_m(x,z)\right)\right|$$

$$= \left|T_f\left(\frac{1}{2\pi i}\int_{[bD_r]^{-1}} \frac{(1-sz)^{-1}}{s-x}ds - \frac{1}{2\pi i}\int_{[bD]^{-1}} \frac{d_m(s,z)}{s-x}ds\right)\right|$$

$$= \left|\frac{1}{2\pi i}\int_{[bD_r]^{-1}} T_f\left(\frac{s^{-1}}{1-xs^{-1}}\right)\left[\frac{1}{1-sz}-d_m(s,z)\right]ds\right|$$

$$\leq \mathcal{L}_f{}^* \sup_{s\in[bD_r]^{-1}}\left|s^{-1}f\left(s^{-1}\right)\right| \sup_{s\in[bD_r]^{-1}}\left|(1-sz)^{-1}-d_m(s,z)\right|$$

$$\leq \mathcal{L}_f{}^* \sup_{s\in[bD_r]^{-1}}\left|(1-sz)^{-1}-d_m(s,z)\right|.$$

Consequently,

$$\sup_{z\in K}\left|f(z)-T_f\left(d_m(x,z)\right)\right| \leq \mathcal{L}_f{}^* \sup_{(s,z)\in[bD_r]^{-1}\times K}\left|(1-sz)^{-1}-d_m(s,z)\right|$$

which completes the *Proof* of the *Lemma*.

Remark 1.1.16. In the *Proof* of *Theorem 1.1.14*, we have used the property that $g(\omega(\mathrm{N}))$ is an open subset of D. To see this, we may proceed as follows: Since $\omega(\mathrm{N}) \supset \mathbb{C}\times\{0\}$, the set $g(\omega(\mathrm{N}))$ contains 0 and therefore it is not empty. Now, fix $z^\circ \in g(\omega(\mathrm{N}))$. Since $\omega(\mathrm{N})$ is a not void open set in $\mathbb{C}^2$, for any $\zeta \in \overline{D}$ there are $\varepsilon(\zeta)>0$ and $\delta(\zeta)>0$ with $(\xi,z)\in\omega(\mathrm{N})$

whenever $|\xi-\zeta|<\varepsilon(\zeta)$ and $|z-z^{\circ}|<\delta(\zeta)$. The set $\overline{D}$ being compact, we can choose $\xi_1,\xi_2,...,\xi_J\in\overline{D}$ such that

$$\overline{D}\subset\bigcup_{j=1}^{J}\Delta(\xi_j,\varepsilon(\xi_j)).$$

By defining

$$s:=\min\{\delta(\xi_j):j=1,2,...,J\},$$

we get $\Delta(z^{\circ},s)\subset g(\omega(\mathrm{N}))$, and hence we have proved that $g(\omega(\mathrm{N}))$ is an open subset of D.

1.2. Approximate Quadrature Formulas for Harmonic Functions

1.2.1. Composed Padé-type Approximation

We begin with the definition of composed Padé-type approximants to a harmonic function $u=u_1+iu_2$ in the unit disk D. Without loss of generality, we shall always assume that $u(0)=0$.

Suppose the restriction to the circle of radius $r<1$ of each real-valued harmonic function u_j has the Fourier representation

$$u_j\left(re^{it}\right)=2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty}\sigma_\nu^{(j)}r^\nu e^{i\nu t}\right)\quad(z=re^{it},-\pi\le t\le\pi,j=1,2),$$

and define the linear functionals

$$T_{u_j}:\mathbb{P}(\mathbb{C})\to\mathbb{C};x^\nu\mapsto\sigma_\nu^{(j)}\quad(j=1,2),$$

where $\mathbb{P}(\mathbb{C})$ is the vector space of all complex analytic polynomials.

Lemma 1.2.1. *Each functional T_{u_j} extends to a linear continuous functional on the vector space $\mathcal{O}(\overline{D})$ of all functions analytic in an open neighborhood of $\overline{D}$ Moreover, there holds*

$$u_j(z) = 2\,\mathrm{Re}\,T_{u_j}\left((1-xz)^{-1}\right) \quad (|z|<1, j=1,2).$$

Proof. Each u_j is the real part of an analytic function, or $u_j = f_j + \overline{f_j}$ where f_j is analytic in D. If

$$f_j(z) = \sum_{\nu=0}^{\infty} \alpha_\nu^{(f)} z^\nu ,$$

then

$$u_j(z) = 2\,\mathrm{Re}\left(f_j(z)\right) = 2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty} \alpha_\nu^{(j)} z^\nu\right).$$

Of course, $\sigma_\nu^{(j)} = \alpha_\nu^{(j)}$ for $\nu \geq 0$. Application of *Cauchy's Integral Formula* shows now that

$$\left|T_{u_j}(p(x))\right| \leq (2\pi)^{-1} \sup_{|s|=r} \left|f_j(s)\right| \sup_{|s|=r^{-1}} \left|p(s)\right|$$

for every $p(x) \in \mathbb{P}(\mathbb{C})$ and $r<1$. By density, there is a continuous extension of T_{u_j} into $\mathcal{O}(\overline{D})$ .In particular, for every fixed point $z \in D$, the number $T_{u_j}\left((1-xz)^{-1}\right)$ is well defined and equals

$$\sum_{\nu=0}^{\infty} \sigma_\nu^{(j)} z^\nu .$$

Hence,

$$u_j(z) = 2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty} \sigma_\nu^{(j)} z^\nu\right) = 2\,\mathrm{Re}\,T_{u_j}\left((1-xz)^{-1} - 1\right)$$

for any $z \in D$. The *Proof* is complete.

If the function $(1-xz)^{-1}$ is replaced by a polynomial $p_j(x,z)$, then $u_j(z)$ is approximated by $2\operatorname{Re}T_{u_j}(p_j(x,z))$, and therefore the function $u(z)$ is approximated by the expression

$$2\operatorname{Re}T_{u_j}(p_1(x,z))+i2\operatorname{Re}T_{u_2}(p_2(x,z)).$$

This is an approximate quadrature formula and leads to a composed Padé-type approximant to the harmonic function $u(z)$.

More precisely, suppose, for each $j=1,2$,

$$\mathrm{M}^{(j)}=\left(\pi_{m,k}^{(j)}\right)_{m\geq 0, 0\leq k\leq m}$$

is any infinite triangular interpolation matrix with $\pi_{m,k}^{(j)}\in\overline{D}$. If $Q_m^{(j)}(x,z)$ denotes the unique Lagrange-Hermite polynomial of degree at most m that interpolates $(1-xz)^{-1}$ in the $(m+1)$ nodes of the m^{th} row of $\mathrm{M}^{(j)}$ (i.e., $Q_m^{(j)}\left(\pi_{m,k}^{(j)},z\right)=\left(1-\pi_{m,k}^{(j)}z\right)^{-1}$, for $k=0,1,2,\ldots,m$), then

Definition 1.2.2. *The complex-valued function*

$$(m/m+1)_u(z)=2\operatorname{Re}T_{u_1}\left(Q_m^{(1)}(x,z)\right)+i2\operatorname{Re}T_{u_2}\left(Q_m^{(2)}(x,z)\right)$$

is called a composed Padé-type approximant to $u(z)$*. The polynomials*

$$V_{m+1}^{(1)}(x)=\gamma_1\prod_{k=0}^{m}\left(x-\pi_{m,k}^{(1)}\right)\ \textit{and}\ V_{m+1}^{(2)}(x)=\gamma_2\prod_{k=0}^{m}\left(x-\pi_{m,k}^{(2)}\right)\ (\gamma_1,\gamma_2\in\mathbb{C}-\{0\})$$

are called the generating polynomials of this approximation.

Remark 1.2.3. If $u(0) \neq 0$, then

$$(m/m+1)_u(z) = 2\operatorname{Re} T_{u_1}\left(Q_m^{(1)}(x.z)\right) i 2\operatorname{Re} T_{u_2}\left(Q_m^{(2)}(x,z)\right) - u(0).$$

Now, put

$$V_{m+1}^{(j)} * (z) := z^{m+1} V_{m+1}^{(j)}(z^{-1}), W_m^{(j)} := T_{u_j}\left(\left[V_{m+1}^{(j)}(x) - V_{m+1}^{(j)}(z)\right] / [x-z]\right)$$

and

$$W_m^{(j)} * (z) := z^m W_m^{(j)}(z^{-1})$$

($j = 1,2$). It is readily seen that

Theorem 1.2.4.(a). $(m/m+1)_u(z)$ *is a complex-valued harmonic function in* D, *with coordinates the real parts of rational functions of type* $(m, m+1)$:

$$(m/m+1)_u(z) = 2\operatorname{Re} T_{u_1}\left(W_m^{(1)} * (z) \big/ V_{m+1}^{(1)} * (z) \right) + i\ 2\operatorname{Re} T_{u_2}\left(W_m^{(2)} * (z) \big/ V_{m+1}^{(2)}(z) \right).$$

(b). *The error of such an approximation equals*

$$(m/m+1)_u(z) - u(z)$$

$$= 2\operatorname{Re}\left[\frac{1}{V_{m+1}^{(1)}(z^{-1})} T_{u_1}\left(\frac{V_{m+1}^{(1)}(x)}{xz-1}\right)\right] + i2\operatorname{Re}\left[\frac{1}{V_{m+1}^{(2)}(z^{-1})} T_{u_2}\left(\frac{V_{m+1}^{(2)}(x)}{xz-1}\right)\right].$$

Proof. It is well known that the general Hermite interpolation polynomial can be deduced from the Lagrange polynomial by continuity arguments when some points coincide. We can therefore assume that the points $\pi_{m,0}^{(j)}, \pi_{m,1}^{(j)}, \ldots, \pi_{m,m}^{(j)}$ are distinct and that $Q_m^{(j)}(x,z)$ is the Lagrange polynomial of degree at most m which interpolates $(1-xz)^{-1}$ in the $(m+1)$ distinct nodes of the

m^{th} row of $\mathrm{M}^{(j)}$. We remind that the Lagrange polynomial of $(1-xz)^{-1}$ at $x=\pi_{m,0}^{(j)},\pi_{m,1}^{(j)},....,\pi_{m,m}^{(j)}$ is given by

$$Q_m^{(j)}(x,z)=\sum_{k=0}^{m}\frac{\left[V_{m+1}^{(j)}(x)-V_{m+1}^{(j)}(\pi_{m,k}^{(j)})\right]\Big/[x-z]}{\left(V_{m+1}^{(j)}\right)'(\pi_{m,k}^{(j)})}\frac{1}{1-\pi_{m,k}^{(j)}z}.$$

By using the definition of $W_m^{(j)}$, we obtain

$$T_{u_j}\left(Q_m^{(j)}(x,z)\right)=\sum_{k=0}^{m}\frac{W_m^{(j)}(\pi_{m,k}^{(j)})}{\left(V_{m+1}^{(j)}\right)'(\pi_{m,k}^{(j)})}\frac{1}{1-\pi_{m,k}^{(j)}z}=z^{-1}\sum_{k=0}^{m}\frac{W_m^{(j)}(\pi_{m,k}^{(j)})}{\left(V_{m+1}^{(j)}\right)'(\pi_{m,k}^{(j)})}\frac{1}{z^{-1}-\pi_{m,k}^{(j)}}.$$

This is the partial fraction decomposition of

$$z^{-1}W_m^{(j)}(z^{-1})\Big/V_{m+1}^{(j)}(z^{-1}).$$

Hence

$$2\operatorname{Re}T_{u_j}\left(Q_m^{(j)}(x,z)\right)=2\operatorname{Re}\left(z^{-1}W_m^{(j)}(z^{-1})\Big/V_{m+1}^{(j)}(z^{-1})\right)=2\operatorname{Re}\left(W_m^{(j)}*(z)\Big/V_{m+1}^{(j)}(z)\right),$$

which completes the *Proof* of *(a)*. To prove*(b)* it suffices to observe that, for any $z\in D$, we have

$$2\operatorname{Re}T_{u_j}\left(Q_m^{(j)}(x,z)\right)-u_j(z)=2\operatorname{Re}\left(W_m^{(j)}*(z)\Big/V_{m+1}^{(j)}*(z)\right)-2\operatorname{Re}T_{u_j}\left((1-xz)^{-1}\right)$$

$$=2\operatorname{Re}\left(\frac{1}{V_{m+1}^{(j)}(z^{-1})}T_{u_j}\left(\frac{V_{m+1}^{(j)}(x)}{x-z^{-1}}\right)-z^{-1}\frac{V_{m+1}^{(j)}(z^{-1})}{V_{m+1}^{(j)}(z^{-1})}T_{u_j}\left((x-z^{-1})^{-1}\right)-T_{u_j}\left((1-xz)^{-1}\right)\right)$$

$$=2\operatorname{Re}\left(\frac{1}{V_{m+1}^{(j)}(z^{-1})}T_{u_u}\left(\frac{V_{m+1}^{(j)}(x)}{xz-1}\right)\right).$$

Let now

$$\mathrm{A}_k^{(j,m)} = \left[W_m^{(j)}(\pi_{m,k}^{(j)}) \Big/ \left(V_{m+1}^{(j)}\right)'\left(\pi_{m,k}^{(j)}\right) \right]$$

for any $k \leq m$. From the *Proof* of *(a)* in the above *Theorem*, it follows that the Newton-Côtes approximate quadrature formula is

$$T_{u_j}\left(Q_m^{(j)}(x,z)\right) = \sum_{k=0}^{m} \mathrm{A}_k^{(j,m)}\left(1-\pi_{m,k}^{(j)}z\right)^{-1} = \sum_{k=0}^{m} \mathrm{A}_k^{(j,m)}\left[1+\pi_{m,k}^{(j)}z+\left(\pi_{m,k}^{(j)}\right)^2 z^2+\ldots\right],$$

that is

$$T_{u_j}\left(Q_m^{(j)}(x,z)\right) = \sum_{\nu=0}^{\infty} d_\nu^{(j,m)} z^\nu \quad \text{with } d_\nu^{(j,m)} = \sum_{k=0}^{m} \mathrm{A}_k^{(j,m)}\left[\pi_{m,k}^{(j)}\right]^\nu.$$

This implies that for any $r<1$, the Fourier series expansion of the restriction $(m/m+1)_{u_r}(t)$ of $(m/m+1)_u(z)$ to the circle of radius r is

$$(m/m+1)_{u_r}(t) = \sum_{\nu=0}^{\infty}\left(d_\nu^{(1,m)}+id_\nu^{(2,m)}\right)r^\nu e^{i\nu t} + \sum_{\nu=0}^{\infty}\left(\overline{d_\nu^{(1,m)}+id_\nu^{(2,m)}}\right)r^\nu e^{-i\nu t}$$

$$= 2\mathrm{Re}\left(\sum_{\nu=0}^{\infty} d_\nu^{(1,m)} r^\nu e^{i\nu t}\right) + i2\mathrm{Re}\left(\sum_{\nu=0}^{\infty} d_\nu^{(2,m)} r^\nu e^{i\nu t}\right) \qquad (,-\pi \leq t \leq \pi).$$

From the exactitude of the Newton-Cotes quadrature formula for polynomials of degree less than m, it follows that

$$d_\nu^{(j,m)} = \sigma_\nu^{(j)} \quad \text{for any } \nu = 0,1,2,\ldots,m \quad (j=1,2).$$

This property justifies the notation composed Padé-type approximant to $u(z)$. Summarizing, we have proved the following crucial property for the composed Padé-type approximation:

Theorem 1.2.5. *The Fourier series expansion of the restriction* $(m/m+1)_{u_r}(t)$ *of* $(m/m+1)_u(z)$ *to any circle of radius* $r<1$ *matches the Fourier series expansion of the restriction* $u_r(t)$ *of* $u(z)$ *to that circle up to the* $\pm m^{th}$ *– order's Fourier term.*

Similarly, we obtain the following more general result:

Theorem 1.2.6. *If*

$$\left|\pi_{m,k}^{(j)}\right| \leq c \ \text{for} \ j=1,2 \ \text{ and any } \ k=0,1,2,\ldots,m,$$

with $c \geq 1$, *the composed Padé-type approximant*

$$(m/m+1)_u(z)$$

to the harmonic complex-valued function $u(z)=u_1(z)+i\,u_2(z)$ *is a harmonic function in the open disk* $D_{1/c}$ *that is centered at* 0 *and has radius* $1/c$. *This harmonic function has coordinates the real parts of two rational functions with denominators* $V_{m+1}^{(1)}*(z)$ *and* $V_{m+1}^{(2)}*(z)$ *and whose numerators*

$$W_m^{(1)}*(z) \ \text{and} \ W_m^{(2)}*(z)$$

are determined by the condition that the Fourier series expansion of the restriction $(m/m+1)_{u_r}(t)$ *of* $(m/m+1)_u(z)$ *to any circle of radius* $r<1/c$ *matches the Fourier series expansion of the restriction* $u_r(t)$ *of* $u(z)$ *to that circle up to the* $\pm m^{th}$ *– order's Fourier term.*

Composed Padé-type approximation is a coordinate procedure. If, instead of a complex-valued harmonic function $u(z)$, we have to approximate in the Padé-type sense a real-valued harmonic function $h(z)$ of D, then there is nothing to change. For emphasis and since in such a case the composed Padé-type approximants are exactly real parts of rational functions of type $(m, m+1)$, we note

$$\mathrm{Re}(m/m+1)_h(z)$$

(instead of $(m/m+1)_u(z)$), and we say that $\mathrm{Re}(m/m+1)_h(z)$ is a *Padé-type approximant* to the real-valued harmonic function $h(z)$. With this notation, we have

$$(m/m+1)_u(z) = \mathrm{Re}(m/m+1)_{u_1}(z) + i\,\mathrm{Re}(m/m+1)_{u_2}(z).$$

Remark 1.2.7. If $f = f_1 + i f_2$ is an analytic function in the disk, with real and imaginary parts the harmonic real-valued functions f_1 and f_2 respectively, then one can show that any Padé-type approximant to f in the classical sense is a composed Padé-type approximant of the form

$$\mathrm{Re}(m/m+1)_{f_1}(z) + i\,\mathrm{Re}(m/m+1)_{f_2}(z) \ .$$

(For a *Proof* see below *Theorem 1.2.15.*)

Let us now turn to Gaussian methods. It is well known that, in Gaussian approximate quadrature formulas, the interpolation points are chosen so that the quadrature formula is exact for polynomials of degree less than $2m+2$. It is also well known that these interpolation points are the roots of orthogonal polynomials.

Let us consider the family of orthogonal polynomials

$$\{q_{m+1}^{(j)}(x) : m = 0,1,2,\ldots\}$$

with respect to the functional T_{u_j}, that is

$$T_{u_j}\left(x^{\nu} q_{m+1}^{(j)}(x)\right) = 0 \text{ for any } \nu = 0,1,2,\ldots,m \quad (j = 1,2).$$

The exact degree of each $q_{m+1}^{(j)}(x)$ is $m+1$. The orthogonality relations are still satisfied if $q_{m+1}^{(j)}(x)$ is multiplied by a constant different from zero. Thus, $q_{m+1}^{(j)}(x)$ is defined apart a multiplying factor. In the sequel, we shall always assume that $q_{m+1}^{(j)}(x)$ is a monic polynomial. A necessary and sufficient condition that $q_{m+1}^{(j)}(x)$ exists uniquely is that the *Hankel determinant*

$$H_{m+1}^{(u_j)}\left(\sigma_0^{(j)}\right) = \det\begin{pmatrix} \sigma_0^{(j)} & \sigma_1^{(j)} & \sigma_2^{(j)} & \cdots & \sigma_m^{(j)} \\ \sigma_1^{(j)} & \sigma_2^{(j)} & \sigma_3^{(j)} & \cdots & \sigma_{m+1}^{(j)} \\ \sigma_2^{(j)} & \sigma_3^{(j)} & \sigma_4^{(j)} & \cdots & \sigma_{m+2}^{(j)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_m^{(j)} & \sigma_{m+1}^{(j)} & \sigma_{m+2}^{(j)} & \cdots & \sigma_{2m}^{(j)} \end{pmatrix}$$

is different from zero. In what follows, we suppose all these determinants are different from zero, that is $H_{m+1}^{(u_j)}\left(\sigma_0^{(j)}\right) \neq 0$ for any $m \neq 0$. In that case the functional T_{u_j} is said to be *definite* and the orthogonal monic polynomials are given by

$$q_{m+1}^{(j)}(x) = \det\begin{pmatrix} \sigma_0^{(j)} & \sigma_1^{(j)} & \sigma_2^{(j)} & \cdots & \sigma_{m+1}^{(j)} \\ \sigma_1^{(j)} & \sigma_2^{(j)} & \sigma_3^{(j)} & \cdots & \sigma_{m+2}^{(j)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma_m^{(j)} & \sigma_{m+1}^{(j)} & \sigma_{m+2}^{(j)} & \cdots & \sigma_{2m+1}^{(j)} \\ 1 & x & x^2 & \cdots & x^{m+1} \end{pmatrix} \Bigg/ H_{m+1}^{(u_j)}\left(\sigma_0^{(j)}\right), \quad m = 0,1,2,\ldots$$

Let us choose the infinite triangular interpolation matrix

$$\mathrm{M}^{(j)} = \left(\pi_{m,k}^{(j)}\right)_{m\geq 0, 0\leq k\leq m}$$

in such a manner that for any m the points $\pi_{m.0}^{(j)}, \pi_{m,1}^{(j)}, ..., \pi_{m,m}^{(j)}$ are the roots of $q_{m+1}^{(j)}(x)$. We can of course consider the Padé-type approximation to $u_j(z)$ with generating polynomial

$$q_{m+1}^{(j)}(x) = \prod_{k=0}^{m}\left(x - \pi_{m,k}^{(j)}\right).$$

Setting

$$q_{m+1}^{(j)} * (x) := x^{m+1} q_{m+1}^{(j)}(x^{-1}), w_m^{(j)}(z) := T_{u_j}\left([q_{m+1}^{(j)}(x) - q_{m+1}^{(j)}(z)] \big/ [x - z] \right),$$

and

$$w_m^{(j)} * (z) := z^m w_m^{(j)}\left(z^{-1}\right),$$

we are convinced that the function

$$2\,\mathrm{Re}\left(w_m^{(j)} * (z) \big/ q_{m+1}^{(j)} * (z) \right) = 2\,\mathrm{Re}\, T_{u_j}\left(Q_m^{(j)}(x,z)\right)$$

is a Padé-type approximant to $u_j(z)$. This approximant has the following strong property:

Theorem 1.2.8. *Assume that all the roots* $\pi_{m,k}^{(j)}$ *of* $q_{m+1}^{(j)}(x)$ *are such that*

$$\left|\pi_{m,k}^{(j)}\right| \le c \text{ for any } k = 0,1,2,...,m,$$

with $c \ge 1$. *If the Fourier series representation of* $2\,\mathrm{Re}\left(w_m^{(j)} * (re^{i\theta}) / q_{m+1}^{(j)} * (re^{i\theta})\right)$ *is*

$$2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty} d_\nu^{(j,m)} r^\nu e^{i\nu\theta}\right) \quad (-\pi \le \theta \le \pi, 0 \le r < \frac{1}{c}),$$

then there holds

$$d_\nu^{(j,m)} = \sigma_\nu^{(j)}$$

for any $\nu = 0,1,2,...,2m+1$.

Proof. Let

$$B_k^{(j,m)} = \left[w_m^{(j)}\left(\pi_{m,k}^{(j)}\right) \Big/ \left(q_{m+1}^{(j)}\right)'\left(\pi_{m,k}^{(j)}\right) \right]$$

for any $k \leq m$. From the *Proof* of *Theorem 1.2.4.(a)*, it follows that

$$T_{u_j}\left(Q_m^{(j)}(x,z)\right) = \sum_{\nu=0}^{\infty} d_\nu^{(j,m)} z^\nu,$$

with

$$d_\nu^{(j,m)} = \sum_{k=0}^{m} B_k^{(j,m)} [\pi_{m,k}^{(j)}]^\nu .$$

This implies that, for any $r < \frac{1}{c}$, the Fourier series expansion of the restriction

$$\mathrm{Re}(m/m+1)_{(u_j)_r}(t)$$

of $\mathrm{Re}(m/m+1)_u(z)$ to the circle of radius r is

$$\mathrm{Re}(m/m+1)_{(u_j)_r}(t) = 2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty} d_\nu^{(j,m)} r^\nu e^{i\nu t}\right).$$

From the exactitude of the Gauss quadrature formula for polynomials of degree less than or equal to $2m+1$, it follows that

$$d_\nu^{(j,m)} = \sigma_\nu^{(j)}$$

for any $\nu = 0,1,2,\ldots,2m+1$.

Corollary 1.2.9. *Assume that, whenever* $j = 1,2$, *all the roots* $\pi_{m,k}^{(j)}$ *of* $q_{m+1}^{(j)}(x)$ *are such that*

$$\left|\pi_{m,k}^{(j)}\right| \leq c \text{ for any } k = 0,1,2,\ldots,m$$

and some constant $c \geq 1$. *The Fourier series expansion of the restriction* $(m/m+1)_{u_r}(t)$ *of* $(m/m+1)_u(z)$ *to any circle of radius* $r < \frac{1}{c}$ *matches the Fourier series expansion of the restriction* $u_r(t)$ *of* $u(z)$ *to that circle up to the* $\pm(2m+1)^{th}$ *Fourier term.*

Motivated by these exactitude results, we give the second basic *Definition* of this *Paragraph*.

Definition 1.2.10. *The function*

$$2\,\mathrm{Re}\left(w_m^{(j)} * (z)/q_{m+1}^{(j)} * (z)\right) = 2\,\mathrm{Re}\,T_{u_j}\left(Q_m^{(j)}(x,z)\right)$$

is called a Padé approximant to the harmonic real-valued function $u_j(z)$. *It will be denoted by*

$$\mathrm{Re}[m/m+1]_{u_j}(z).$$

The function

$$\mathrm{Re}[m/m+1]_{u_1}(z) + i\,\mathrm{Re}[m/m+1]_{u_2}(z)$$

is called a composed Padé approximant to the harmonic complex-valued function $u(z) = u_1(z) + i\,u_2(z)$. *It will be denoted by*

$$[m/m+1]_u(z).$$

We have the following result characterizing Padé approximation to harmonic real-valued functions:

Theorem 1.2.11.([42]) **(a).** *If*

$$H_{m+1}^{(u_j)}\left(c_0^{(j)}\right) \neq 0 \quad ,$$

the Padé approximant $\mathrm{Re}[m/m+1]_{u_j}(z)$ *to the harmonic real-valued function* $u_j(z)$ *is uniquely determined, in the sense that there is no other real part of complex rational function with the property described in Theorem 1.2.8.*

(b). *If all the roots* $\pi_{m,k}^{(j)}$ *of* $q_{m+1}^{(j)}(x)$ *are such that*

$$\left|\pi_{m,k}^{(j)}\right| \leq c \text{ for any } k \leq m,$$

with $c \geq 1$, *then the error of the Padé approximation to* $u_j(z)$ $(|z| < 1)$ *is given by*

$$\mathrm{Re}[m/m+1]_{u_j}(z) - u_j(z)$$

$$= 2\,\mathrm{Re}\left[\frac{z^{2m+2}}{q_{m+1}^{(j)}*(z)} T_{u_j}\left(\frac{x^{m+1}q_{m+1}^{(j)}(x)}{xz-1}\right)\right] = 2\,\mathrm{Re}\left[\frac{z^{2m+2}}{\left(q_{m+1}^{(j)}*(z)\right)^2} T_{u_j}\left(\frac{\left(q_{m+1}^{(j)}\right)^2(x)}{xz-1}\right)\right].$$

Of course, this *Theorem* generalizes immediately to the context of composed Padé approximation.

Let us finally turn to the construction of Padé or Padé -type approximants to $u_j(z)$ as real parts of rational functions with arbitrary degrees in the numerator and denominator. For any $n \geq 1$, $u_j(z)$ is rewritten as

$$u_j(z) = 2\,\mathrm{Re}\left(\sum_{\nu=0}^{n-1} \sigma_\nu^{(j)} z^\nu + z^n u_j^{(n)}(z)\right) = 2\,\mathrm{Re}\left(\sum_{\nu=0}^{n-1} \sigma_\nu^{(j)} z^\nu + z^n \sum_{\nu=0}^{\infty} \sigma_{n+\nu}^{(j)} z^\nu\right).$$

If $Q_m^{(j)}(x,z)$ is the interpolation polynomial of $(1-xz)^{-1}$ at the $(m+1)$ zeroes $\pi_{m,0}^{(j)}, \pi_{m,1}^{(j)}, \ldots, \pi_{m,m}^{(j)}$ of the polynomial $q_{m+1}^{(j)}(x)$ (respectively, of the generating polynomial $V_{m+1}^{(j)}(x)$), the function

$$2\,\mathrm{Re}\left(\sum_{\nu=0}^{n-1} \sigma_\nu^{(j)} z^\nu + z^n T_{u_j^{(n)}}\left(Q_m^{(j)}(x,z)\right)\right)$$

is the real part of a rational fraction of type $(m+n, m+1)$. If $\left|\pi_{m,k}^{(j)}\right| \leq c$ for any $k \leq m$, with $c \geq 1$, and if z is replaced by

$$re^{i\theta}\left(, r < \frac{1}{c}\right),$$

then the Fourier series expansion of this function, with respect to the variable $\theta \in [-\pi, \pi]$, matches the Fourier representation of $u_j\left(re^{i\theta}\right)$ up to the $\pm(2m+n+1)^{th}$ Fourier term (respectively, up to the $\pm(m+n)^{th}$ Fourier term). This function is said to be a Padé approximant to $u(z)$ of higher order and it is denoted by

$$\mathrm{Re}[m+n/m+1]_{u_j}(z).$$

(Respectively, this function is said to be a Padé-type approximant to $u(z)$ of higher order and it is denoted by

$$\mathrm{Re}(m+n/m+1)_{u_j}(z)\,).$$

The function

$$\mathrm{Re}[m+n/m+1]_{u_1}(z)+i\,\mathrm{Re}[m+n/m+1]_{u_2}(z)$$

is called a *composed Padé approximant to* $u(z)=u_1(z)+i\,u_2(z)$ *of higher order* and is denoted by

$$[m+n/m+1]_u(z).$$

Similarly, the function

$$\mathrm{Re}(m+n/m+1)_{u_1}(z)+i\,\mathrm{Re}(m+n/m+1)_{u_2}(z)\,,$$

is a *composed Padé -type approximant to* $u(z)=u_1(z)+i\,u_2(z)$ *of higher order* and is simply denoted by

$$(m+n/m+1)_u(z).$$

1.2.2 Convergence Results

In studying Padé-type approximation to analytic functions in the disk, one problem of considerable interest were that of describing the suitable choice of the generating polynomials in order to establish the convergence of the corresponding sequence of Padé-type approximants. The purpose of this *Paragraph* is to study the same question about Padé-type approximants to harmonic functions. The techniques used are similar to those proposed by Eiermann in [55].

First, suppose u is a harmonic real-valued function in the open unit disk with $u(0) = 0$. (Otherwise, we consider the difference $U = u - u(0)$.) Then $u(z)$ can be written as

$$u(z) = 2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty} \sigma_\nu z^\nu\right) \quad (|z| < 1).$$

As we have seen, if $T_u : \mathbb{P}(\mathbb{C}) \to \mathbb{C}$ is the linear functional defined by $T_u(x^\nu) = \sigma^\nu$ (ν =0,1,2,...) then the function $u(z)$ can be rewritten in the form

$$u(z) = 2\,\mathrm{Re}\, T_u\left((1 - xz)^{-1}\right) \quad (|z| < 1).$$

Given now an infinite triangular interpolation matrix

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m \geq 0, 0 \leq k \leq m}$$

with complex entries $|\pi_{m,k}| \leq 1$, for any $z \in \mathbb{C} - \{\pi_{m,k}^{-1} : k = 0,1,\ldots,m\}$ let $Q_m(x,z)$ denotes the unique polynomial of degree at most m which interpolates $(1 - xz)^{-1}$ in the $(m+1)$ nodes of the m^{th} row of M. If some of the nodes $\pi_{m,k}$ coincide, the interpolation has to be understood in the Hermite sense. It is obvious that each polynomial $Q_m(x,z)$ can be expressed in the form

$$Q_m(x,z) = \sum_{k=0}^{m} \beta_{m,k}(z) \sum_{\nu=0}^{k} x^\nu z^\nu,$$

where $\beta_{m,k}(z)$ are complex-valued functions in $z \in \mathbb{C} - \{\pi_{m,k}^{-1} : k = 0,1,\ldots,m\}$. Thus, if

$$\mathrm{N}(z) = \left(\beta_{m,k}(z)\right)_{m \geq 0, 0 \leq k \leq m}$$

then the summability method to the function

$$u(z) = 2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty} \sigma_\nu z^\nu\right)$$

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induced by $\mathrm{N}(z)$ is a sequence of Padé-type approximants to $u(z)$:

$$\{\mathrm{Re}(m/m+1)_u(z) = 2\,\mathrm{Re}\,T_u(Q_m(x,z)) = 2\,\mathrm{Re}\left(\sum_{k=0}^{m}\beta_{m,k}(z)\sum_{\nu=0}^{k}\sigma_\nu z^\nu\right):$$

$$z \in \mathbb{C} - \{\pi_{m,k}^{-1} : k \le m\}, m = 0,1,2,...\}.$$

With this notation, the convergence behaviour of the sequence

$$\{u(z) - \mathrm{Re}(m/m+1)_u(z) : m = 0,1,2,...\}$$

depends on the convergence of the sequence

$$\{(1-xz)^{-1} - \sum_{k=0}^{m}\beta_{m,k}(z)\sum_{\nu=0}^{k}x^\nu z^\nu : m = 0,1,2,...\}:$$

the generalized version of Okada's Theorem as given by Eiermann in [55] *implies that if* $\mathbb{C}\times\{0\}$ *is contained into an open set* $\omega(\mathrm{N}) \subset \mathbb{C}^2$ *into which the sequence*

$$\{\sum_{k=0}^{m}\beta_{m,k}(z)\sum_{\nu=0}^{k}x^\nu z^\nu : m = 0,1,2,...\}$$

converges compactly to $(1-xz)^{-1}$, *then*

$$\lim_{m\to\infty}\mathrm{Re}(m/m+1)_u(z) = u(z)$$

compactly on

$$g(\omega(\mathrm{N})) = \{z \in D : (\zeta, z) \in \omega(\mathrm{N}), |\zeta| \le 1\}.$$

Since

$$(1-xz)^{-1} - \sum_{k=0}^{m}\beta_{m,k}(z)\sum_{\nu=0}^{k}x^\nu z^\nu = (1-xz)^{-1} - Q_m(x,z),$$

and since the interpolation polynomial of $(1-xz)^{-1}$ satisfies

$$Q_m(x,z) = (1-xz)^{-1}\left(1 - \frac{V_{m+1}(x)}{V_{m+1}(z^{-1})}\right) \quad ([23]),$$

we have proved the

Theorem 1.2.12. *If the generating polynomials*

$$V_{m+1}(x)=\gamma\prod_{k=0}^{m}(x-\pi_{m,k})$$

satisfy

$$\lim_{m\leftrightarrow\infty}\frac{V_{m+1}(x)}{V_{m+1}(z^{-1})}=0$$

compactly in an open set $\omega\subset\mathbb{C}^2$ *containing* $\mathbb{C}\times\{0\}$, *then there holds*

$$\lim_{m\to\infty}\mathrm{Re}(m/m+1)_u(z)=u(z)$$

compactly in $\{z\in D:(\zeta,z)\in\omega,|\zeta|\leq 1\}$, *for any* $u(z)$ *harmonic real-valued function in the disk.*

Just as we did for the *Proof* of *Theorem 1.2.12*, we can verify the following convergence result for a sequence of composed Padé-type approximants.

Theorem 1.2.13. *Let* $u(z)$ *be a harmonic complex-valued function in the open unit disk* D. *If the generating polynomials*

$$V_{m+1}^{(1)}(x)=\gamma_1\prod_{k=0}^{m}\left(x-\pi_{m,k}^{(1)}\right)\text{ and }V_{m+1}^{(2)}(x)=\gamma_2\prod_{k=0}^{m}\left(x-\pi_{m,k}^{(2)}\right)$$

satisfy

$$\lim_{m\to\infty}\frac{V_{m+1}^{(1)}(x)}{V_{m+1}^{(1)}(z^{-1})}=\lim_{m\to\infty}\frac{V_{m+1}^{(2)}(x)}{V_{m+1}^{(2)}(z^{-1})}=0$$

compactly into an open subset ω of $\mathbb{C}^2$ *containing* $\mathbb{C}\times\{0\}$, *then*

$$\lim_{m\to\infty}\mathrm{Re}(m/m+1)_u(z)=u(z)$$

compactly in $\{z\in D:(\zeta,z)\in\omega,|\zeta|\leq 1\}$.

Let us mention three direct and interesting applications of *Theorem 1.2.13*.

Corollary 1.2.14. *Let* $u(z)$ *be a harmonic complex-valued function of* D.

(a). *If the generating polynomials are*

$$V_{m+1}^{(1)}(x) = (x-\alpha)^{m+1} \;\; and \;\; V_{m+1}^{(2)}(x) = (x-\beta)^{m+1} \quad (\alpha, \beta \in \mathbb{C},\ m = 0,1,2,\ldots),$$

then the corresponding sequence $\{(m/m+1)_u(z) : m = 0,1,2,\ldots\}$ *of composed Padé-type approximants to* u *converges to* $u(z)$ *compactly in the open set*

$$\Omega = \{z \in D : \left|z^{-1}-\alpha\right| > \sup_{|\xi|<1}\left|\xi-\alpha\right| \;\; and \;\; \left|z^{-1}-\beta\right| > \sup_{|\xi|<1}\left|\xi-\beta\right|\}.$$

(b). *If the generating polynomials are*

$$V_{m+1}^{(1)}(x) = \prod_{k=0}^{m}\left(x-\pi_k^{(1)}\right) \;\; and \;\; V_{m+1}^{(2)}(x) = \prod_{k=0}^{m}\left(x-\pi_k^{(2)}\right)$$

($m = 0,1,2,\ldots$), *where each sequence* $\{\pi_k^{(j)} : k = 0,1,2,\ldots\}$ *has* $N^{(j)}$ *limit points* $L_0^{(j)}, L_1^{(j)}, \ldots, L_{N^{(j)}-1}^{(j)}$ *approached cyclically* (i.e. $\lim_{n\to\infty}\pi_{n,N+k}^{(j)} = L_k^{(j)}$), *then the corresponding sequence* $\{(m/m+1)_u(z) : m = 0,1,2,\ldots\}$ *of composed Padé-type approximants to* u *converges to* $u(z)$ *compactly on*

$$\Lambda = \{z \in D : z^{-1} \notin \left(\overline{\Theta_{p_1}^{(1)}}\right) \bigcup \left(\overline{\Theta_{p_2}^{(2)}}\right)\},$$

where $\Theta_{p_j}^{(j)}$ *is the lemniscate with focci* $L_0^{(j)}, L_1^{(j)}, \ldots, L_{N^{(j)}-1}^{(j)}$ *and radius*

$$p_j = \sup_{|\xi|\le 1}\left|\prod_{k=0}^{N^{(j)}-1}\left(\xi - L_k^{(j)}\right)\right|.$$

(c). *If the generating polynomials* $V_{m+1}^{(j)}(x)$ *are the Tchebycheff polynomials*

$$V_{m+1}^{(1)}(x) = V_{m+1}^{(2)}(x) = \prod_{k=0}^{m}\left(x - \cos\left[\frac{2k+1}{2m+1}\pi\right]\right) \quad (m = 0,1,2,...),$$

then the corresponding sequence $\{(m/m+1)_u(z) : m = 0,1,2,...\}$ *of composed Padé-type approximants to* u *converges to* $u(z)$ *compactly into the open set*

$$\Lambda = \{z \in D : |z^{-1}+1| + |z^{-1}-1| > \sup_{|\xi|\le 1}(|\xi+1| + |\xi-1|)\}.$$

1.2.3. Connection with the Classical Theory

To confirm the coherence of our theory on Padé-type approximation to Fourier series, we are indented to explain its consistency with the classical one about Padé-type approximation to analytic functions. The aim of this *Paragraph* is to certify this coherence, by showing that *classical Padé-type approximants are a special case of composed Padé-type approximants: if* $f = f_1 + i f_2$ *is any analytic function in the disk, with respective real and imaginary parts the harmonic real-valued functions* f_1 *and* f_2, *then every Padé-type approximant* $(m/m+1)_f$ *to* f *is a composed Padé-type approximant to* f *of the form*

$$\mathrm{Re}(m/m+1)_{f_1} + i\,\mathrm{Re}(m/m+1)_{f_2}.$$

Suppose $f = f_1 + i\, f_2$ is analytic in the open disk:

$$f(z) = \sum_{\nu=0}^{\infty} \alpha_\nu^{(f)} z^\nu \quad (|z| < 1).$$

Assume that $f(0) = \alpha_0^{(f)} = 0$. Otherwise, we may consider the difference $f - f(0)$. If we restrict f to any circle of fixed radius $r < 1$, we obtain a continuous function on that circle which can also be interpreted as a function on the unit circle:

$$f_r(t) = f\left(r\, e^{i\,t}\right).$$

Now, observe that

$$f_r(t) = \sum_{\nu=0}^{\infty} \alpha_\nu^{(f)} r^\nu e^{i\nu t}$$

This in particular means that the ν^{th} Fourier coefficient of f_r is $\alpha_\nu^{(f)} r^\nu$ for $\nu \geq 0$, and is zero for $\nu < 0$.

On the other hand, the function f being harmonic, the restriction of each part f_j to the circle of radius r has a Fourier representation:

$$\left[f_j\right]_r(t) = f_j\left(re^{it}\right) = 2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty} c_\nu^{(f)} r^\nu e^{i\nu t}\right) \qquad (j = 1,2).$$

Hence

$$\sum_{\nu=0}^{\infty} \alpha_\nu^{(f)} r^\nu e^{i\nu t} = 2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty} c_\nu^{(f_1)} r^\nu e^{i\nu t}\right) + i2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty} c_\nu^{(f_2)} r^\nu e^{i\nu t}\right)$$

$$= \sum_{\nu=0}^{\infty}\left(c_\nu^{(f_1)} + ic_\nu^{(f_2)}\right) r^\nu e^{i\nu t} + \sum_{\nu=0}^{\infty}\left(\overline{c_\nu^{(f_1)} - ic_\nu^{(f_2)}}\right) r^\nu e^{-i\nu t},$$

or, after a change of variables,

$$\sum_{\nu=0}^{\infty} \alpha_\nu^{(f)} z^\nu = \sum_{\nu=0}^{\infty}\left(c_\nu^{(f_1)} + ic_\nu^{(f_2)}\right) z^\nu + \sum_{\nu=0}^{\infty}\left(\overline{c_\nu^{(f_1)} - ic_\nu^{(f_2)}}\right) \overline{z}^\nu \qquad (z \in D).$$

From the analyticity of f, we get

$$\alpha_\nu^{(f)} = c_\nu^{(f_1)} + ic_\nu^{(f_2)} \quad \text{and} \quad c_\nu^{(f_1)} = ic_\nu^{(f_2)}$$

for any $\nu \geq 0$. (Or, if one wishes, one may deduce these equations directly from *Cauchy's Integral Formula*

$$\alpha_\nu^{(f)} = \frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)}{\zeta^{\nu+1}}d\zeta = \frac{1}{2\pi}\frac{1}{r^\nu}\int_{-\pi}^{\pi}\frac{f_r(t)}{e^{i\nu t}}dt\,,$$

and, from the fact that

$$c_\nu^{(f_j)} = \frac{1}{2\pi}\frac{1}{r^\nu}\int_{-\pi}^{\pi}\frac{[f_j](t)}{e^{i\nu t}}dt\,.$$

Indeed, it is immediately seen that

$$\alpha_\nu^{(f)} = \frac{1}{2\pi r^\nu}\int_{-\pi}^{\pi}\frac{f_r(t)}{e^{i\nu t}}dt = \frac{1}{2\pi r^\nu}\int_{-\pi}^{\pi}\frac{[f_1]_r(t)}{e^{i\nu t}}dt + i\frac{1}{2\pi r^\nu}\int_{-\pi}^{\pi}\frac{[f_2]_r(t)}{e^{i\nu t}}dt = c_\nu^{(f_1)} + ic_\nu^{(f_2)}$$

and consequently,

$$c_\nu^{(f_1)} - ic_\nu^{(f_2)} = 0\text{ ,for any } \nu \geq 0.)$$

Define now the linear functionals

$$T_f : \mathbf{P}(\mathbf{C}) \to \mathbf{C} : x^\nu \mapsto T_f(x^\nu) := \alpha_\nu^{(f)},$$

$$T_{f_1} : \mathbf{P}(\mathbf{C}) \to \mathbf{C} : x^\nu \mapsto T_{f_1}(x^\nu) := c_\nu^{(f_1)},$$

and

$$T_{f_2} : \mathbf{P}(\mathbf{C}) \to \mathbf{C} : x^\nu \mapsto T_{f_2}(x^\nu) := c_\nu^{(f_2)}.$$

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It is well known that these functionals extend continuously and linearly to the common larger vector space $\mathcal{O}(\overline{D})$ (see *Lemma 1.2.1*). Further, since

$$\alpha_\nu^{(f)} = c_\nu^{(f_1)} + ic_\nu^{(f_2)} \quad \text{and} \quad c_\nu^{(f_1)} = ic_\nu^{(f_2)}$$

for any $\nu \geq 0$, we also have

$$\alpha_\nu^{(f)} = \left(c_\nu^{(f_1)} + ic_\nu^{(f_2)}\right) + \left(\overline{c_\nu^{(f_1)} - ic_\nu^{(f_2)}}\right) = 2\operatorname{Re} c_\nu^{(f_1)} + i2\operatorname{Re} c_\nu^{(f_2)}$$

and therefore

$$T_f\left(x^\nu\right) = 2\operatorname{Re} T_{f_1}\left(x^\nu\right) + i2\operatorname{Re} T_{f_2}(x^\nu) \quad \text{for all} \quad \nu \geq 0.$$

By linearity and density, we obtain

$$T_f(g) = 2\operatorname{Re} T_{f_1}(g) + i2\operatorname{Re} T_{f_2}(g) \qquad \text{for any } g \in \mathcal{O}(\overline{D}).$$

In particular, for any fixed $z \in D$, it holds

$$T_f\left(\frac{1}{1-xz}\right) = 2\operatorname{Re} T_{f_1}\left(\frac{1}{1-xz}\right) + i2\operatorname{Re} T_{f_2}\left(\frac{1}{1-xz}\right)$$

(where T_f, T_{f_1}, T_{f_2} act on the variable x, while z is regarded as a parameter).

Choosing

$$\mathrm{M} = \mathrm{M}^{(1)} = \mathrm{M}^{(2)} = \left(\pi_{m,k}\right)_{m \geq 0, 0 \leq k \leq m},$$

with $\pi_{m,k} \in \mathbb{C}$, and letting, for any $m \geq 0$ and $z \in D - \{\pi_{m,k}^{-1} : k \leq m\}$, $Q_m(x,z)$ be the unique polynomial which interpolates $(1-xz)^{-1}$ at $x = \pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}$ it is clear that

$$T_f\left(Q_m(x,z)\right) = 2\operatorname{Re} T_{f_1}\left(Q_m(x,z)\right) + i2\operatorname{Re} T_{f_2}\left(Q_m(x,z)\right)$$

and hence we have proved the

Theorem 1.2.15. ([44]) *Every Padé-type approximant to a function analytic in the unit disk is a composed Padé-type approximant to this function.*

1.2.4. Numerical Examples

Example 1.2.16. Let D be the open unit disk of C and let $f : D \to$ R be the harmonic function

$$f(z) = \operatorname{Re} z.$$

We will consider several different cases:

(a). Choose $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = \pi_{3,3} = 0$. Then

$$\operatorname{Re}(3/4)_f(z) = 2\operatorname{Re}\left[\frac{1/2z^3}{1/z^4}\right] = \operatorname{Re} z = f(z).$$

(b). Choose m=3 and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = \pi_{3,3} = \frac{1}{4}$. Then

$$\operatorname{Re}(3/4)_f(z) = 2\operatorname{Re}\frac{16z^3 - 128z^2 + 128z}{z^4 - 16z^3 + 32z^2 - 256z + 256}.$$

Thus,

(c). Choose $m = 3$ and $\pi_{3,0} = -1, \pi_{3,1} = -\frac{1}{2}, \pi_{3,2} = -\frac{1}{3}, \pi_{3,3} = -\frac{1}{4}$. Then,

$$\operatorname{Re}(3/4)_f(z) = 2 \cdot \operatorname{Re}\left[\frac{\frac{24}{48}z + \frac{50}{48}z^2 + \frac{35}{48}z^3}{1 + \frac{50}{24}z + \frac{35}{24}z^2 + \frac{10}{24}z^3 + \frac{1}{24}z^4}\right].$$

Thus,

(d). Choose $m = 3$ and

z	$\operatorname{Re}(3/4)_f(z)$	$f(z) = \operatorname{Re} z$
0	0.0000000	0
$\frac{1}{2}$	0.4888747	0.5
$\frac{1}{2} + i\frac{1}{2}$	0.5463334	0.5
$-i\frac{3}{4}$	0.0132866	0

$$\pi_{3,k} = \cos\left(\frac{2k+1}{7}\pi\right) : \pi_{3,0} = \cos\frac{\pi}{7} = \cos(25.714285^{o}) = 0.9009688,$$

$$\pi_{3,1} = \cos\left(\frac{3\pi}{7}\right) = \cos(77.142857^{o}) = 0.2225209,$$

$$\pi_{3,2} = \cos\left(\frac{5\pi}{7}\right) = \cos(128.57142^{o}) = -0.6234898,$$

$$\pi_{3,3} = \cos\left(\frac{7\pi}{7}\right) = \cos(180^{o}) = -1.$$

Then,

$$\mathrm{Re}(3/4)_f(z) = 2\,\mathrm{Re}\left[\frac{4z+2z^2-4z^3}{8+4z-8z^2-3z^3+z^4}\right].$$

Thus,

z	$\mathrm{Re}(3/4)_f(z)$	$f(z) = \mathrm{Re}\,z$
0	0.0000000	0
$\frac{1}{2}$	0.4888888	0.5
$\frac{1}{2}+i\cdot\frac{1}{2}$	0.5305536	0.5
$-i\cdot\frac{3}{4}$	-0.0094428	0

Example 1.2.17. Let D be the open unit planar disk and let $f : D \to \mathbb{R}$ be the harmonic function

$$f(z) = \operatorname{Im} z\,.$$

As in the *Example 1.2.16*, we will consider several and different cases:

(a). Choosing $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = \pi_{3,3} = 0$, we have

$$\operatorname{Re}(3/4)_f(z) = 2 \cdot \operatorname{Re}\left[\frac{\frac{-i}{2z^3}}{\frac{1}{z^4}}\right] = \operatorname{Re}(-iz) = \operatorname{Im} z = f(z)\ .$$

(b). If we choose $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = \pi_{3,3} = \frac{1}{4}$, then

$$\operatorname{Re}(3/4)_f(z) = 2 \cdot \operatorname{Re}\left[\frac{-i\,16z^3 + i128\,z^2 - i\,128\,z}{z^4 - 16\,z^3 + 32\,z^2 - 256\,z + 256}\right]$$

and

z	$\operatorname{Re}(3/4)_f(z)$	$f(z) = \operatorname{Re} z$
0	0.000	0.0
$\frac{1}{2}$	0.520	0.5
$\frac{1}{2} + i\,\frac{1}{2}$	0.406	0.5
$-i\,\frac{3}{4}$	0.150	0.0
$\frac{4}{5} - i\,\frac{1}{2}$	0.280	0.8

$$\mathrm{Re}(3/4)_f(z) = 2 \cdot \mathrm{Re}\left[\frac{-i\,16z^3 + i128\,z^2 - i\,128\,z}{z^4 - 16\,z^3 + 32\,z^2 - 256\,z + 256}\right]$$

and

z	$\mathrm{Re}(3/4)_f(z)$	$f(z) = \mathrm{Re}\,z$
0	0.000000	0.00
$\frac{1}{2}$	0.0000000	0.00
$\frac{1}{2} + i\,\frac{1}{2}$	0.4859469	0.50
$-i\,\frac{3}{4}$	-0.7596724	-0.75

(c). If $m = 3$ and $\pi_{3,0} = -1, \pi_{3,1} = -\frac{1}{2}, \pi_{3,2} = -\frac{1}{3}, \pi_{3,3} = -\frac{1}{4}$, then

$$\mathrm{Re}(3/4)_f(z) = 2 \cdot \mathrm{Re}\left[\frac{-i\,\frac{24}{48}\,z - i\,\frac{50}{48}\,z^2 - i\,\frac{35}{48}\,z^3}{1 + \frac{50}{24}\,z + \frac{35}{24}\,z^2 + \frac{10}{24}\,z^3 + \frac{1}{24}\,z^4}\right]$$

and

z	$\mathrm{Re}(3/4)_f(z)$	$f(z)$
0	0.000000	0.00
$\frac{1}{2}$	0.0000000	0.00
$\frac{1}{2}+i\frac{1}{2}$	0.5432119	0.50
$-i\frac{3}{4}$	-0.7112400	-0.75

(d). If $m=3$ and

$$\pi_{3,k}=\cos\left(\frac{2k+1}{7}\pi\right):\pi_{3,0}=\cos\frac{\pi}{7}=0.9009688,$$

$$\pi_{3,1}=\cos\frac{3\pi}{7}=0.2225209,$$

$$\pi_{3,2}=\cos\frac{5\pi}{7}=-0.6234898,$$

$$\pi_{3,3}=\cos\pi=-1,$$

then

$$\mathrm{Re}(3/4)_f(z)=2\,\mathrm{Re}\left[\frac{4z+2z^2-4z^3}{8+4z-8z^2-3z^3+z^4}\right],$$

and

z	$\mathrm{Re}(3/4)_f(z)$	$f(z)$
0	0.0000000	0.00
$\frac{1}{2}$	0.0000000	0.00
$\frac{1}{2}+i\,\frac{1}{2}$	0.4965407	0.50
$-i\,\frac{3}{4}$	-0.7111234	0.75

Example 1.2.18. Let f and g be the two real-valued harmonic functions

$$f : D \to \mathbb{R};\ z \mapsto f(x) = \mathrm{Re}\frac{1}{1-z} \text{ and } g : D \to \mathbb{R};\ z \mapsto g(z) = \mathrm{Im}\frac{1}{1-z}$$

$\left(D = \{z \in \mathbb{C} : |z| < 1\}\right)$.

(a). For $m = 5$ and $\pi_{5,0} = \pi_{5,1} = \pi_{5,2} = \pi_{5,3} = \pi_{5,4} = \pi_{5,5} = 0$, we have

$$\mathrm{Re}(5/6)_f(z) = 2\ \mathrm{Re}\frac{1}{2}\left(2 + z + z^2 + z^3 + z^4 + z^5\right) - 1 = \mathrm{Re}\left(1 + z^2 + z^3 + z^4 + z^5\right)$$

and

$$\mathrm{Re}(5/6)_g(z) = 2\ \mathrm{Re}\left[\frac{(-i)}{2}\left(z + z^2 + z^3 + z^4 + z^5\right)\right] = \mathrm{Re}\left[(-i)\left(z + z^2 + z^3 + z^4 + z^5\right)\right]$$

Indicatively, we can state the following numerical results:

z	$f(z)$	$\mathrm{Re}(5/6)_f(z)$	$g(z)$	$\mathrm{Re}(5/6)_g(z)$
0	1.0000000	1.0000000	0.0000000	0.0000000
$\frac{i}{5}$	0.9615384	0.9573333	0.1923076	0.1872000
$\frac{i}{3}-i\frac{3}{5}$	0.8287292	0.750209	-0.7458563	-0.6596859
$\frac{1}{8}$	1.1428571	1.1440298	0.0000000	0.0000000

(b). For $m=4$ and $\pi_{4,0}=-1, \pi_{4,1}=-\frac{1}{3}, \pi_{4,2}=-\frac{1}{9}, \pi_{4,3}=-\frac{1}{27}, \pi_{4,4}=-\frac{1}{81}$, we have

$$\mathrm{Re}(4/5)_f(z)=\mathrm{Re}\frac{183798\,z^4+187188\,z^3+212598\,z^2+235467\,z+118098}{59049+88209\,z+32670\,z^2+3630\,z^3+120\,z^4+z^5}-1,$$

$$\mathrm{Re}(4/5)_g(z)=\mathrm{Re}\frac{183798\,z^4+179928\,z^3+147258\,z^2+59049\,z}{i\left(59049+88209\,z+32670\,z^2+3630\,z^3+120\,z^4+z^5\right)}$$

and

z	$f(z)$	$\mathrm{Re}(4/5)_f(z)$	$g(z)$	$\mathrm{Re}(4/5)_g(z)$
0	1.0000000	1.0000000	0.0000000	0.0000000
$\frac{i}{3}-i\frac{3}{5}$	0.8287292	0.6467485	-0.7458563	-0.9878018
$\frac{i}{2}$	0.8000000	0.7814009	-0.4000000	0.3255728
$\frac{1}{6}$	1.2000000	1.1996231	0.0000000	0.0000000

(c). If $m=3$ and $\pi_{3,0}=\pi_{3,1}=\pi_{3,2}=0, \pi_{3,3}=1$, then

$$\operatorname{Re}(3/4)_f(z)=\operatorname{Re}\frac{2-z}{1-z}-1 \quad \text{and} \quad \operatorname{Re}(3/4)_g(z)=\operatorname{Re}\frac{z}{i-iz} .$$

The efficacy of these approximations becomes apparent in the next table:

z	$f(z)$	$\operatorname{Re}(3/4)_f(z)$	$g(z)$	$\operatorname{Re}(3/4)_g(z)$
0	1.0000000	1.0000000	0.0000000	0.0000000
$\frac{i}{3}-i\frac{3}{5}$	0.8287292	0.8287292	-0.7458563	-0.5469613
$\frac{4}{5}i$	0.6097560	0.6097560	0.4878048	0.4878048
$\frac{1}{7}$	1.1666666	1.1666666	0.0000000	0.0000000

Example 1.2.19. We will now approximate in the Padé-type sense the real-valued function

$$f(z)=\log|1-z| \qquad (z\in D).$$

(a). If $m=5$ and $\pi_{5,0}=\pi_{5,1}=\pi_{5,2}=\pi_{5,3}=\pi_{5,4}=\pi_{5,5}=0$, then

$$\operatorname{Re}(5/6)_f(z)=-\operatorname{Re}\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\frac{z^5}{5}\right)$$

and

z	$f(z)=\log\lvert 1-z\rvert$	$\mathrm{Re}(5/6)_f(z)$
0	0.0000000	0.0000000
$\frac{i}{2}$	-0.6931471	-0.6272916
$-\frac{5}{6}$	0.6061358	0.5777392
$-i\frac{3}{4}$	0.2231435	0.2021484
$\frac{4}{5}-i\frac{1}{2}$	-0.6189371	-0.7032277

(b). For $m=3$ and $\pi_{3,0}=\pi_{3,1}=\pi_{3,2}=0, \pi_{3,3}=1$, we have

$$\mathrm{Re}(3/4)_f(z)=\frac{1}{6}\,\mathrm{Re}\frac{z^3+3z^2-6z}{1-z}.$$

Indicatively, one has

z	$f(z)$	$\mathrm{Re}(4/5)_f(z)$
0	0.0000000	0.0000000
$\frac{1}{4}$	-0.287682	-0.2881944
$-\frac{5}{6}$	0.6061358	0.6123737
$i\frac{2}{7}$	0.0784716	0.0387626
$\frac{i}{2}+i\frac{1}{2}$	-0.3465735	-0.3333333
$\frac{4}{5}-i\frac{1}{2}$	-0.6189371	-0.494885

Example 1.2.20. Let f be the real-valued harmonic function

$$f : D \to \mathbb{R}\ ;\ z \mapsto f(z) = \log\left|\frac{1+z}{1-z}\right|.$$

D is always the open unit planar disk.

(a). For $m = 6$ and $\pi_{6,0} = \pi_{6,1} = \pi_{6,2} = \pi_{6,3} = \pi_{6,4} = \pi_{6,5} = \pi_{6,6} = 0$, it holds

$$\operatorname{Re}(6/7)_f(z) = 2\operatorname{Re}\left(\frac{z^{-1}\left(\frac{z^{-1}}{5} + \frac{z^{-3}}{3} + z^{-5}\right)}{z^{-7}}\right) = \frac{2}{15}\operatorname{Re}\left(3z^5 + 5z^3 + 15z\right).$$

Thus

z	$f(z) = \log\left\lvert 1+z/1-z \right\rvert$	$\operatorname{Re}(6/7)_f(z)$
0	0.0000000	0.000000
$\frac{1}{7}$	0.2876820	0.2876816
$i\frac{\sqrt{3}}{2}$	0.0000000	0.0000000
$-i\frac{\sqrt{3}}{2}$	0.0000000	0.0000000
$\frac{6}{7}$	2.5649494	2.1076896
$-\frac{6}{7}$	-2.5649494	-2.1076896
$\frac{1}{3}-i\frac{3}{5}$	0.4886851	0.4860707
$\frac{4}{5}-i\frac{1}{2}$	1.243888	1.2604053

(b). For $m=3$ and $\pi_{3,k}=\cos\left(\frac{2k+1}{7}\pi\right)$ $(k=0,1,2,3)$, we have

$$\mathrm{Re}(3/4)_f(z)=2\ \mathrm{Re}\left[\frac{z+0.5\,z^2-\frac{2}{3}\,z^3}{1+0.5\,z-z^2-0.375\,z^3+0.125\,z^4}\right].$$

In the following table, we have collected some trivial cases:

z	$f(z)=\log\left\lvert 1+z/1-z\right\rvert$	$\mathrm{Re}(3/4)_f(z)$
0	0.0000000	0.0000000
$\frac{1}{2}$	1.0986123	1.1273712
$\frac{i}{5}$	0.0000000	-0.0371954
$\frac{1}{2}+i\frac{1}{2}$	0.8047189	0.7328022
$\frac{1}{8}+i\frac{4}{9}$	0.2090678	0.2146718

1.3. Padé and Padé-Type Approximation to L^p-Functions

1.3.1. Preliminaries

In this *Paragraph*, we put the preparatory material which we shall need in the sequel. Since the text is expository the proofs will be omitted; they can be found in the literature.

The original *Dirichlet Problem* was the following one: *given a real-valued continuous function f on the unit circle* C, *find a continuous function on the closed disk* $\overline{D}$ *which agrees with f on the circle and which is harmonic in the open disk* D. This *Problem* is completely solved by the *Poisson Integral Formula*: if P_r is the real-valued function whose Fourier coefficients are $r^{|\nu|}$, i.e.

$$P_r(\theta) = \sum_{-\infty}^{\infty} r^{|\nu|} e^{i\nu\theta} = \mathrm{Re}\left[\frac{1+re^{i\theta}}{1-re^{i\theta}}\right] = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

$(0 \le r < 1, -\pi \le \theta \le \pi)$, then the family

$$\{P_r : 0 \le r < 1\}$$

is called *the Poisson kernel of the unit disk* and *any harmonic function* u *in the open disk has the following integral representation*

$$u\left(re^{i\theta}\right) = u_r(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)P_r(\theta - t)dt \quad (0 \le r < 1, -\pi \le \theta \le \pi).$$

Theorem 1.3.1. *Let* u *be a complex-valued function in* L^p *of the unit circle, where* $1 \le p < \infty$. *Define* u *in the unit disk by*

$$u\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)P_r(\theta - t)dt\ .$$

Then the extended function u is harmonic in the open unit disk, and, as $r \to 1$, *the functions* $u_r(\theta) = u\left(re^{i\theta}\right)$ *converge to* u *in the* $L^p -$*norm. If* u *is continuous on the unit circle, the* u_r *converge uniformly to* u; *thus, the extended* u *is continuous on the closed disk, harmonic in the interior*.

Theorem 1.3.2. *Let μ be a finite complex Baire measure on the unit circle and let*

$$u\left(re^{i\theta}\right)=\int_{-\pi}^{\pi}P_r(\theta-t)d\mu(t).$$

Then u is harmonic in the open disk and the measures

$$d\mu_r=\frac{1}{2\pi}u_r(\theta)d\theta$$

converge to μ in the weak-star topology on measures.

In all the above cases, it would seem convenient to say that the harmonic function $u\left(re^{i\theta}\right)$is the *Poisson integral* of the corresponding function or measure on the unit circle. This will save us some words as we proceed to reverse the process. We will now ask: given a harmonic function in the disk, how do we ascertain if it is the Poisson integral of some type of function or measure on the unit circle? If u is harmonic, then

$$u\left(re^{i\theta}\right)=\sum_{-\infty}^{\infty}c_\nu r^{|\nu|}e^{i\nu\theta}$$

so the question is actually: when is $\{c_\nu : \nu = 0,\pm1,\pm2,...\}$ the sequence of Fourier coefficients of some type of function or measure? We have the following

Theorem 1.3.3. *Let u be a complex-valued harmonic function in the open unit disk. Write*

$$u_r(\theta)=u\left(re^{i\theta}\right).$$

(a). *u is the Poisson integral of an L^1 −function on the unit circle if and only if the u_r converge in the L^1-norm.*

(b). *If $1<p\le\infty$, then u is the Poisson integral of an L^p −function on the unit circle if and only if the functions u_r are bounded in L^p − norm.*

(c). *u is the Poisson integral of a continuous function on the unit circle if and only if the u_r converge uniformly.*

(d). *u is the Poisson integral of a finite complex Baire measure on the unit circle if and only if the u_r are bounded in $L^1 -$norm.*

(e). *u is the Poisson integral of a finite positive Baire measure on the unit circle if and only if u is non-negative.*

The $L^{\infty} -$part of *(b)* is *Fatou's Theorem*. The interesting part of it is the fact that any bounded harmonic function in the disk is the Poisson integral of a bounded Baire function on the unit circle. Part *(e)* is *Herglotz's Theorem*: every non-negative harmonic function is the Poisson integral of a positive measure. One should note that in any of the cases above the harmonic function u is real-valued if and only if the corresponding $L^p -$function or measure is real. For more information, we wish to note that the Poisson integral of a Lebesgue-integrable function u in the unit circle has a radial limit $\lim_{r\to 1} u\left(re^{i\theta}\right)$ at almost every point of the circle, and this limit is almost everywhere equal to u. More generally, we might state the following:

Theorem 1.3.4. *Let u be a complex-valued harmonic function in the open unit disk. Suppose that the integrals*

$$\int_{-\pi}^{\pi} \left|u\left(re^{i\varsigma}\right)\right|^{p} d\theta$$

are bounded as $r \to 1$ for some p, $1 \le p < \infty$. Then for almost every θ the radial limits

$$\tilde{u} = \lim_{r\to 1} u\left(re^{i\theta}\right)$$

exist and define a function $\tilde{u}$ in L^p of the circle. Moreover,

(a). *if $p > 1$, then u is the Poisson integral of $\tilde{u}$;*

(b). *if* $p=1$, *then* u *is the Poisson integral of a unique finite measure whose absolutely continuous part is*

$$\frac{1}{2\pi}\tilde{u}\,d\theta.$$

If u *is a bounded harmonic function, the boundary values exist almost everywhere and define a bounded measurable function* ? *whose Poisson integral is* u.

For more about the *Dirichlet Problem* and the various boundary value results see Bieberbach [11], Courant [36], Evans [57], Hoffman [81], Lasser [91] and Zygmund [150].

1.3.2. Composed Padé-Type Approximation to L^p- Functions

A large number of the harmonic case properties still hold for other classes of functions and measures. Among others, we will first discuss the case of real-valued L^p-functions in the unit circle C.

Let again

$$\mathrm{M}=\left(\pi_{m,k}\right)_{m\geq 0,0\leq k\leq m}$$

be an infinite triangular interpolation matrix with complex entries

$$\pi_{m,k}\in D.$$

For any fixed $z\in\mathbb{C}-\{\pi_{m,k}^{-1}:k=0,1,...,m\}$, let $Q_m(x,z)$ denote the unique polynomial of degree at most m, which interpolates $(1-xz)^{-1}$ in the $(m+1)$ nodes of the m^{th} row of M, i.e.

$$Q_m(\pi_{m,k}, z) = (1 - \pi_{m,k} z)^{-1}$$

for any $k \le m$.

Let also $u(e^{it})$ be a real-valued function in $L^p(C)$, $1 \le p \le \infty$ $(-\pi \le t \le \pi)$, with sequence of Fourier coefficients $\{\sigma_\nu : \nu = 0, \pm 1, \pm 2, \ldots\}$. Define the Poisson integral of $u(e^{it})$ by

$$u_r(t) = u(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta}) P_r(t-\theta) d\theta = \sum_{\nu=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta}) e^{-i\nu\theta} d\theta\right) r^{|\nu|} e^{i\nu t}$$
$$= \sum_{\nu=-\infty}^{\infty} \sigma_\nu r^{|\nu|} e^{i\nu t}$$

($0 \le r < 1$ and $\{P_r(\cdot)\}$ is the Poisson kernel of D). From the solution of *Dirichlet*'s *Problem* in D, it follows that the extended function $u(re^{it}) = u(z)$ is harmonic in the open unit disk and satisfies

$$\lim_{r \to 1} \left\| u_r(t) - u(e^{it}) \right\|_p = 0.$$

For any $m \ge 0$, we can therefore consider the Padé-type approximant

$$\mathrm{Re}(m/m+1)_u(z) = 2\,\mathrm{Re}\,T_u(Q_m(x,z)) - \sigma_0$$

to $u(z)$, where T_u is the linear functional on the space of complex polynomials defined by

$$T_u(x^\nu) = \sigma_\nu \quad (\nu = 0,1,2,\ldots).$$

As it is pointed out in *Theorem 1.2.6*, this approximant is a harmonic real-valued function in D, with Fourier series expansion

$$\mathrm{Re}(m/m+1)_u(re^{it}) = \sum_{\nu=0}^{\infty} d_\nu^{(m)} r^\nu e^{i\nu t} + \sum_{\nu=0}^{\infty} \overline{d_\nu^{(m)}} r^\nu e^{-i\nu t}$$

($0 \le r < 1, -\pi \le t \le \pi$), such that

$d_\nu^{(m)} = \frac{r^{-\nu}}{2\pi}\int_{-\pi}^{\pi} u(re^{it})e^{-i\nu\theta}d\theta = \sigma_\nu$ and $\overline{d_\nu^{(m)}} = \frac{r^{-\nu}}{2\pi}\int_{-\pi}^{\pi} u(re^{i\theta})e^{i\nu\theta}d\theta = \sigma_{-\nu}$ ($0 \le \nu \le m$). Since it is assumed that $|\pi_{m,k}| < 1$ for any $k \le m$, it is easily verified that the approximant $\mathrm{Re}(m/m+1)_u(z)$ can be extended continuously on the closed unit disk. In fact, according to *Theorem 1.2.4.(a)*, we have

$$\mathrm{Re}(m/m+1)_u(re^{it}) = 2\,\mathrm{Re}\left(W_m{}^*(re^{it}) \middle/ V_{m+1}{}^*(re^{it}) \right) - \sigma_0$$

and it suffices to consider the radial limits

$$\mathrm{Re}(m/m+1)_u(e^{it}) = 2\,\mathrm{Re}\,T_u(Q_m(x,z)) - \sigma_0 = 2\,\mathrm{Re}\left(W_m{}^*(e^{it}) \middle/ V_{m+1}{}^*(e^{it}) \right) - \sigma_0$$

$$-\pi \le t \le \pi.$$

We are now in position to give the definition of Padé-type approximants to L^p – functions.

Definition 1.3.5. *Let*

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m}(x - \pi_{m,k})$$

be a generating polynomial, with $\gamma \in \mathbb{C} - \{0\}$ and $|\pi_{m,k}| < 1$ for any $k \le m$.

Let u be a real-valued function in $L^p(C)$, $1 \le p \le \infty$. Suppose

$$\sum_{\nu=-\infty}^{\infty} \sigma_\nu e^{i\nu t}$$

is the Fourier series expansion of $u(e^{it})$. (Notice that, in general,

$$u(e^{it}) \neq \sum_{\nu=-\infty}^{\infty} \sigma_\nu e^{i\nu t}.)$$

Consider the linear functional

$$T_u : \mathbb{P}(\mathbb{C}) \to \mathbb{C} : T_u(x^\nu) = \sigma_\nu ,$$

and define

$$V_{m+1}{}^*(x) := x^{m+1} V_{m+1}(x^{-1}), \quad W_m(z) := T_u\left([V_{m+1}(z) - V_{m+1}(x)] \big/ [z - x] \right),$$

$$W_m{}^*(z) := z^m W_m(z^{-1}).$$

The function

$$\mathrm{Re}(m/m+1)_u(z) = 2\,\mathrm{Re}\left(W_m{}^*(z) \big/ V_{m+1}{}^*(z) \right) - \sigma_0 = 2\,\mathrm{Re}\, T_u(Q_m(x,z)) - \sigma_0 .$$

($|z| < 1$) *is continuous everywhere on* C, *and is said to be a Padé-type approximant to* $u(z)$.

Mention that each $u(z) \in L^p(C)$ can also be viewed as a function $\hat{u}(t) \in L^p[-\pi, \pi]$, with

$$\hat{u}(t) = u(e^{it}) \quad \text{and} \quad \hat{u}(-\pi) = \hat{u}(\pi)$$

and with the same sequence of Fourier coefficients

$$\{\sigma_\nu : \nu = 0, \pm 1, \pm 2, \ldots\}.$$

Conversely, any function f in $L^p[-\pi,\pi]$, which is 2π – periodic (i.e. $f(-\pi) = f(\pi)$) is identified with a function $f(e^{it}) = f(z) \in L^p(C)$. Hence, via the identification $L^p(C)$ $\equiv \{f \in L^p[-\pi,\pi] : f$ *is a* 2π*-periodic*$\}$, we can introduce Padé-type approximation to every real-valued function $f \in L^p[-\pi,\pi]$ such that $f(-\pi) = f(\pi)$.

Definition 1.3.6. *Let*

$$V_{m+1}(x)=\gamma\prod_{k=0}^{m}\left(x-\pi_{m,k}\right)$$

be a generating polynomial, with $\gamma\in\mathbb{C}-\{0\}$ *and such that* $\left|\pi_{m,k}\right|<1$ *for* $k=0,1,2,\ldots,m$.

Let f *be a real-valued* $2\pi-$*periodic function in* $L^{p}[-\pi,\pi]$, $1\le p\le\infty$. *Suppose* $\sum_{\nu=-\infty}^{\infty}c_{\nu}e^{i\nu t}$ *is the Fourier series representation of* $f(t)$. (Of course, in general

$$f(t)\neq\sum_{\nu=-\infty}^{\infty}c_{\nu}e^{i\nu t}\,.)$$

Consider the linear functional

$$T_{f}:\mathbb{P}(\mathbb{C})\to\mathbb{C}:T_{f}\left(x^{\nu}\right)=c_{\nu}$$

associated with f, and define

$$V_{m+1}*(x):=x^{m+1}V_{m+1}\left(x^{-1}\right),\quad W_{m+1}(z):=T_{f}\left(\left[V_{m+1}(z)-V_{m+1}(x)\right]/\left[z-x\right]\right),\quad W_{m}*(z):=z^{m}W_{m}\left(z^{-1}\right).$$

The function

$$\mathrm{Re}(m/m+1)_{f}(t)=2\,\mathrm{Re}\left(W_{m}*\left(e^{it}\right)/V_{m+1}*\left(e^{it}\right)\right)-c_{0}=2\,\mathrm{Re}\,T_{f}\left(Q_{m}\left(x,e^{it}\right)\right)-c_{0}$$

$(-\pi\le t\le\pi)$ *is continuous everywhere on* $[-\pi,\pi]$, *and is called a Padé-type approximant to* $f(t)$.

Thus, if σ_0=0,then a Padé-type approximant to $u(z)$ ($|z|=1$) is the real-part of a rational fraction of type $(m,m+1)$, with respect to the variable z. A Padé-type approximant to $f(t)$ ($-\pi \le t \le \pi$) is the real part of a rational function of type $(m,m+1)$, with respect to the dependent variable $s = s(t) = e^{it}$, only if $c_0 = 0$.

Furthermore, the above two definitions mean that *if only a few Fourier coefficients* σ_ν (respectively, c_ν) *of the real-valued functions* $u(z) \in L^p(C)$ (respectively, *of the real-valued* $2\pi -$*periodic function* $f(t) \in L^p[-\pi,\pi]$) *are known, then one can approximate* $u(z)$ (respectively, $f(t)$) *by an approximant in the Padé-type sense*. The crucial property, which justifies the notation Padé-type approximant is described in the following:

Theorem 1.3.7. (a). *If* $|\pi_{m,k}| < 1$ *for any* $k \le m$ *and if the Fourier series expansion of* $\mathrm{Re}(m/m+1)_u(e^{it})$ *is*

$$\sum_{\nu=-\infty}^{\infty} d_\nu^{(m)} e^{i\nu t},$$

then

$$d_\nu^{(m)} = \sigma_\nu$$

for every $\nu = 0, \pm 1, \pm 2, \ldots$.

(b). *If* $|\pi_{m,k}| < 1$ *for any* $k \le m$ *and if the Fourier series expansion of* $\mathrm{Re}(m/m+1)_f(t)$ *is*

$$\sum_{\nu=-\infty}^{\infty} \beta_\nu^{(m)} e^{i\nu t},$$

then

$$\beta_\nu^{(m)} = c_\nu$$

whenever $\nu = 0, \pm 1, \ldots, \pm m$.

Proof. (a). Since every harmonic real-valued function in the unit disk, with continuous boundary values, is the Poisson integral of its continuous restriction to the unit circle, we have

$$\mathrm{Re}(m/m+1)_u\left(re^{it}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Re}(m/m+1)_u\left(e^{i\theta}\right)P_r(t-\theta)d\theta$$

and henceforth the Fourier series expansion of $\mathrm{Re}(m/m+1)_u\left(re^{it}\right)$ is given by

$$\mathrm{Re}(m/m+1)_u\left(re^{it}\right)=\sum_{\nu=-\infty}^{\infty}d_\nu^{(m)}r^{|\nu|}e^{i\nu t} \qquad (0\le r<1, -\pi\le t\le\pi).$$

From *Theorem 1.2.5*, it follows that

$$d_\nu^{(m)}=\sigma_\nu$$

for any $\nu=0,\pm1,\pm2,\ldots,\pm m$.

*(b)***.** As it is mentioned above, if $f(t)\in L^p[-\pi,\pi]$ is a $2\pi-$periodic real-valued function, there is a function $u(z)\in L^p(C)$ having the same Fourier coefficients with f, and such that $u\left(e^{it}\right)=f(t)$ for all $t\in[-\pi,\pi]$. Repetition of the same argument as in *(a)* shows that the Fourier coefficients of $\mathrm{Re}(m/m+1)_u\left(e^{it}\right)$ coincide with the Fourier coefficients of $u\left(e^{it}\right)$ up to the $\pm m^{th}$ order term. We thus conclude that also the Fourier coefficients of $\mathrm{Re}(m/m+1)_f(t)$ coincide with the Fourier coefficients of $f(t)$ up to the $\pm m^{th}$ order term.

The errors of these approximations are given by the following theoretical formulas:

Theorem 1.3.8.(a).

$$\mathrm{Re}(m/m+1)_u\left(e^{it}\right)-u\left(e^{it}\right)=\lim_{r\to 1}2\,\mathrm{Re}\left[\frac{1}{V_{m+1}\left(r^{-1}e^{-it}\right)}T_u\left(\frac{V_{m+1}(x)}{xre^{it}-1}\right)\right],$$

in the L^p-*norm.*

(b).

$$\mathrm{Re}(m/m+1)_f(t)-f(t)=\lim_{r\to 1}2\,\mathrm{Re}\left[\frac{1}{V_{m+1}\left(r^{-1}e^{-it}\right)}T_f\left(\frac{V_{m+1}(x)}{xre^{it}-1}\right)\right],$$

in the L^p-*norm.*

Proof. Let $\{0<r_n<1: n=0,1,2,...\}$ be any sequence satisfying $\lim_{n\to\infty}r_n=1$. It is well known that

$$\lim_{n\to\infty}\left\|u\left(r_ne^{it}\right)-u\left(e^{it}\right)\right\|_p=0.$$

By the uniform convergence of the sequence $\left\{\mathrm{Re}(m/m+1)_u\left(r_n e^{it}\right): n=0,1,2,...\right\}$ to the radial limit $\mathrm{Re}(m/m+1)_u\left(e^{it}\right)$, we also get

$$\lim_{n\to\infty}\left\|\mathrm{Re}(m/m+1)_u\left(r_ne^{it}\right)-\mathrm{Re}(m/m+1)_u\left(e^{it}\right)\right\|_p=0.$$

Letting $\varepsilon>0$, it is clear that there exists a $N=N(\varepsilon)$ with

$$\left\|u\left(r_ne^{it}\right)-u\left(e^{it}\right)\right\|_p<\frac{\varepsilon}{2}\text{ and }\left\|\mathrm{Re}(m/m+1)_u\left(r_ne^{it}\right)-\mathrm{Re}(m/m+1)_u\left(e^{it}\right)\right\|_p<\frac{\varepsilon}{2},$$

for any $n\geq N$. It follows that

$$\left\|u\left(r_ne^{it}\right)-u\left(e^{it}\right)\right\|_p+\left\|\mathrm{Re}(m/m+1)_u\left(r_ne^{it}\right)-\mathrm{Re}(m/m+1)_u\left(e^{it}\right)\right\|_p<\varepsilon,$$

for $n\geq N$, and therefore

$$\left\| \left[u\left(r_n e^{it}\right) - \mathrm{Re}(m/m+1)_u \left(r_n e^{it}\right)\right] - \left[u\left(e^{it}\right) - \mathrm{Re}(m/m+1)_u \left(e^{it}\right)\right] \right\|_p < \varepsilon ,$$

for $n \geq N$, or, by *Theorem 1.2.4.(b)*,

$$\left\| \left[\mathrm{Re}(m/m+1)_u \left(e^{it}\right) - u\left(e^{it}\right)\right] - 2\,\mathrm{Re}\left[\frac{1}{V_{m+1}\left(r_n^{-1} e^{-it}\right)} T_u \left(\frac{V_{m+1}(x)}{x r_n e^{it} - 1} \right) \right] \right\|_p < \varepsilon ,$$

for any $n \geq N$. This proves the first assertion of the *Theorem*. To prove the second assertion, it suffices to use the standard identification of $\left\{ f \in L^p[-\pi,\pi] : f \text{ is } 2\pi - \text{periodic} \right\}$ with $L^p(C)$ and repeat the same argument.

One should contrast this result with the situation for analytic functions.

The continuous case is much more precise.

Theorem 1.3.8. (a). *If* u *is a real-valued continuous function on* C, *then*

$$\mathrm{Re}(m/m+1)_u \left(e^{it}\right) - u\left(e^{it}\right) = \lim_{r \to 1} 2\,\mathrm{Re}\left[\frac{1}{V_{m+1}\left(r^{-1} e^{-it}\right)} T_u \left(\frac{V_{m+1}(x)}{x r e^{it} - 1} \right) \right],$$

uniformly on $[-\pi,\pi]$.

(b). *If* f *is a real-valued* $2\pi -$ *periodic continuous function on* $[-\pi,\pi]$, *then*

$$\mathrm{Re}(m/m+1)_f (t) - f(t) = \lim_{r \to 1} 2\,\mathrm{Re}\left[\frac{1}{V_{m+1}\left(r^{-1} e^{-it}\right)} T_f \left(\frac{V_{m+1}(x)}{x r e^{it} - 1} \right) \right],$$

uniformly on $[-\pi,\pi]$.

The *Proof* of *Theorem 1.3.8* is similar to that of *Theorem 1.3.7*.

We can also give a more concrete description for the theoretical expressions stealing in the error formulas of *Theorems 1.3.8* and *1.3.9*.

Proposition 1.3.10. (a). *If* $u \in L^p(C)$ *is real-valued* ($1 \le p \le \infty$), *then there holds*

$$T_u\left(\frac{V_{m+1}(x)}{xre^{it}-1}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{u(e^{is})}{re^{i(t-s)}-1}V_{m+1}(e^{-is})\,ds \quad (0 \le r < 1, -\pi \le t \le \pi).$$

(b). *If* $f \in L^p[-\pi,\pi]$ *is* $2\pi-$*periodic and real-valued* ($1 \le p \le \infty$), *then there holds*

$$T_f\left(\frac{V_{m+1}(x)}{xre^{it}-1}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(s)}{re^{i(t-s)}-1}V_{m+1}(e^{-is})\,ds \quad (0 \le r < 1 \text{ and } -\pi \le t \le \pi).$$

Proof. Let $0 \le r < 1, -\pi \le t \le \pi$. Set

$$V_{m+1}(x)=\sum_{k=0}^{m} b_k^{(m)}x^k .$$

Application of the continuity property for the linear functionals T_u and T_f shows that

$$T_u\left(\frac{V_{m+1}(x)}{xre^{it}-1}\right)=-\sum_{\nu=0}^{\infty}r^\nu e^{i\nu t}\sum_{k=0}^{m}\frac{b_k^{(m)}}{2\pi}\int_{-\pi}^{\pi}u(e^{is})e^{-i(\nu+k)s}ds$$

and

$$T_f\left(\frac{V_{m+1}(x)}{xre^{it}-1}\right)=-\sum_{\nu=0}^{\infty}r^\nu e^{i\nu t}\sum_{k=0}^{m}\frac{b_k^{(m)}}{2\pi}\int_{-\pi}^{\pi}f(s)e^{-i(\nu+k)s}ds.$$

Computing, we obtain

$$T_u\left(\frac{V_{m+1}(x)}{xre^{it}-1}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(e^{is})\left\{-\left(\sum_{k=0}^{m}b_k^{(m)}e^{-iks}\right)\left(\sum_{\nu=0}^{\infty}r^\nu e^{i(t-s)\nu}\right)\right\}ds$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(e^{is})\left\{\frac{V_{m+1}(e^{-is})}{re^{i(t-s)}-1}\right\}ds.$$

Similarly, one shows that

$$T_f\left(\frac{V_{m+1}(x)}{xre^{it}-1}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(s)\left\{\frac{V_{m+1}\left(e^{-is}\right)}{re^{i(t-s)}-1}\right\}ds,$$

and the *Proposition* follows.

Next, we shall see how to construct Padé-type approximants to continuous functions as real parts of rational functions with arbitrary degrees in the numerator and the denominator.

Let again $u(z)=u\left(e^{it}\right)$ be a real-valued continuous function on the unit circle C, with Fourier series expansion

$$\sum_{\nu=-\infty}^{\infty}\sigma_\nu e^{i\nu t}.$$

The Poisson integral of u is then a harmonic real-valued function in the unit disk D, defined by

$$u_r(t)=u\left(re^{it}\right)=\sum_{\nu=-\infty}^{\infty}\sigma_\nu r^{|\nu|}e^{i\nu t}\quad (0\le r<1).$$

For any $n\ge 0$, $u\left(re^{it}\right)$ can also be written in the form

$$u\left(re^{it}\right)=2\operatorname{Re}T_u\left(\left(1-xre^{it}\right)^{-1}\right)=2\operatorname{Re}\left(\sum_{\nu=0}^{n-1}\sigma_\nu r^\nu e^{i\nu t}+r^n e^{int}u_n\left(re^{it}\right)\right)$$

$$=2\operatorname{Re}\left(\sum_{\nu=0}^{n-1}\sigma_\nu r^\nu e^{i\nu t}+r^n e^{int}T_{u_n}\left[\left(1-xre^{it}\right)^{-1}\right]\right),$$

where we have used the notation $u_n\left(re^{it}\right)$ for the series

$$\sum_{\nu=0}^{\infty}\sigma_{n+\nu}r^\nu e^{i\nu t},$$

and where we have considered the linear functional

$$T_{u_n}:\mathbb{P}(\mathbb{C})\to\mathbb{C}:x^\nu\mapsto T_{u_n}\left(x^\nu\right)=\sigma_{n+\nu}.$$

It is now clear that the Padé-type approximant to $u\left(re^{it}\right)$

$$2\operatorname{Re}(m+n/m+1)_u\left(re^{it}\right)=2\operatorname{Re}\left(\sum_{\nu=0}^{n-1}\sigma_\nu r^\nu e^{i\nu t}+r^n e^{int}T_{u_n}\left(Q_m\left(x,re^{it}\right)\right)\right)-\sigma_0$$

($0\le r<1,-\pi\le t\le\pi$) has the following uniform radial limit as $r\to 1$:

$$2\operatorname{Re}\left(\sum_{\nu=0}^{n-1}\sigma_\nu e^{i\nu t}+e^{int}T_{u_n}\left(Q_m\left(x,e^{it}\right)\right)\right)-\sigma_0$$

$$=\lim_{r\to 1}2\operatorname{Re}\left(\sum_{\nu=0}^{n-1}\sigma_\nu r^\nu e^{i\nu t}+r^n e^{int}T_{u_n}\left(Q_m\left(x,re^{it}\right)\right)\right)-\sigma_0\quad(-\pi\le t\le\pi).$$

If we set $z=e^{it}$, this limit can also be written as

$$2\operatorname{Re}\left(\sum_{\nu=0}^{n-1}\sigma_\nu z^\nu+z^nT_{u_n}\left(Q_m(x,z)\right)\right)-\sigma_0\quad(|z|=1).$$

This is the real part of a rational function of type $(m+n,m+1)$ in z. *If*

$$|\pi_{m,k}|<1\ ,$$

then the Fourier series expansion of the above limit matches the Fourier series expansion of $u\left(re^{it}\right)$ *up to the terms of* $\pm(m+n)^{th}$ *order.* Motivated by this property, we shall say that this limit is again a *Padé-type approximant to* $u(z)$ ($|z|=1$). So, we can write

$$\operatorname{Re}(m+n/m+1)_u(z):=2\operatorname{Re}\left(\sum_{\nu=0}^{n-1}\sigma_\nu z^\nu+z^nT_{u_n}\left(Q_m(x,z)\right)\right)-\sigma_0\quad(|z|=1).$$

Similarly, given a continuous $2\pi-$periodic function

$$f:[-\pi,\pi]\to\mathrm{R}:t\mapsto f(t)$$

with Fourier series representation

$$\sum_{\nu=-\infty}^{\infty}c_\nu e^{i\nu t},$$

the function

$$\mathrm{Re}(m+n/m+1)_f(t) := 2\,\mathrm{Re}\left(\sum_{\nu=0}^{n-1} c_\nu e^{i\nu t} + e^{int} T_{f_n}\left(Q_m\left(x, e^{it}\right)\right)\right) - c_0$$

($t \in [-\pi, \pi]$) is the real part of a rational function of type $(m+n, m+1)$ with respect to the dependent variable $s = s(t) = e^{it}$ and is called *a Padé-type approximant to* $f(t)$. The functional T_{f_n} :P(C)$\to$C is now defined by

$$T_{f_n}\left(x^\nu\right) = c_{n+\nu}\,.$$

The fundamental property of such an approximant is the coincidence of its Fourier representation with that of f *up to the* $(\pm m+n)^{th}$ *Fourier term.*

If instead of a harmonic complex-valued function we have to approximate

a complex-valued function which is in $L^p(C)$ $(1 \le p \le \infty)$,

either

a complex-valued $2\pi-$periodic function in $L^p[-\pi, \pi]$ $(1 \le p \le \infty)$,

then composed Padé-type approximation can be defined analogously.

Indeed, as before, let us consider two infinite triangular interpolation matrices

$$\mathrm{M}^{(1)} = \left(\pi_{m,k}^{(1)}\right)_{m\ge 0, 0\le k\le m} \text{ and } \mathrm{M}^{(2)} = \left(\pi_{m,k}^{(2)}\right)_{m\ge 0, 0\le k\le m}.$$

In contrast to the harmonic complex case, where we have supposed $\left|\pi_{m,k}^{(j)}\right| \le c$ $(|c| \ge 1)$, we will now assume that

$$\pi_{m,k}^{(j)} \in \mathrm{C} \text{ for any } m, k \text{ and any } j = 1,2.$$

Let again $Q_m^{(j)}(x,z)$ be the unique polynomial of degree at most m which interpolates $(1-xz)^{-1}$ in the $(m+1)$ nodes of the m^{th} row of $\mathrm{M}^{(j)}$, i.e.

$$Q_m^{(j)}\left(\pi_{m,k}^{(j)}, z\right) = \left(1 - \pi_{m,k}^{(j)} z\right)^{-1}, \text{ for } k = 0,1,2,\ldots,m \ (j = 1,2).$$

If $u = u_1 + iu_2$ is any complex-valued function in $L^p(C)$, $1 \le p \le \infty$, we define the Poisson integral of $u(z)$ by setting

$$u_j\left(re^{i\theta}\right)=2\,\mathrm{Re}\left(\sum_{\nu=0}^{\infty}\sigma_\nu^{(j)}r^\nu e^{i\nu t}\right)-\sigma_0^{(j)}\quad(0\le r<1,-\pi\le t\le\pi\ \text{ and }\ j=1,2),$$

where $\sigma_0^{(j)},\sigma_{\pm1}^{(j)},\sigma_{\pm2}^{(j)},\ldots$ are the Fourier coefficients of $u_j\left(e^{it}\right)$. If we introduce the linear functionals

$$T_{u_j}:\mathbb{P}(\mathbb{C})\to\mathbb{C}:x^\nu\mapsto T_{u_j}\left(x^\nu\right)=\sigma_\nu^{(j)},$$

then each continuous function

$$\mathrm{Re}(m/m+1)_{u_j}\left(e^{it}\right)=2\,\mathrm{Re}\,T_{u_j}\left(Q_m^{(j)}\left(x,e^{it}\right)\right)-\sigma_0^{(j)}\quad(j=1,2)$$

is a Padé-type approximant to $u_j\left(e^{it}\right)$, with generating polynomial

$$V_{m+1}^{(j)}(x)=\gamma_j\prod_{k=0}^{m}\left(x-\pi_{m,k}^{(j)}\right)\quad(\gamma_j\in\mathbb{C}-\{0\}).$$

Definition 1.3.11. *The complex-valued function*

$$(m/m+1)_u(z):=\mathrm{Re}(m/m+1)_{u_1}(z)+i\,\mathrm{Re}(m/m+1)_{u_2}(z)$$

$$=2\,\mathrm{Re}\,T_{u_1}\left(Q_m^{(1)}(x,z)\right)+i2\,\mathrm{Re}\,T_{u_2}\left(Q_m^{(2)}(x,z)\right)-\left[\sigma_0^{(1)}+i\sigma_0^{(2)}\right]\quad(|z|=1)$$ *is said to be a composed Padé-type approximant to* $u(z)$.

It is easily verified that, if $\sigma_0^{(1)}+i\sigma_0^{(2)}=0$, then $(m/m+1)_u(z)$ is a complex-valued continuous function on the unit circle C, with coordinates the real parts of rational functions of type $(m,m+1)$ with respect to the variable z. In fact, putting

$$V_{m+1}^{(j)}*:=x^{m+1}V_{m+1}^{(j)}\left(x^{-1}\right)$$

$$W_m^{(j)}(z):=T_{u_j}\left(\frac{V_{m+1}^{(j)}(x)-V_{m+1}^{(j)}(z)}{x-z}\right)$$

and

$$W_m^{(j)}*(z) := z^m W_m^{(j)}\left(z^{-1}\right)$$

($j = 1,2$) , we obtain

$$(m/m+1)_u(z) = 2\operatorname{Re}\frac{W_m^{(1)}*(z)}{V_{m+1}^{(1)}*(z)} + i2\operatorname{Re}\frac{W_m^{(2)}*(z)}{V_{m+1}^{(2)}*(z)}.$$

Similarly, via the identification

$$L^p(C) \equiv \{f \in L^p[-\pi,\pi] : f \text{ is } 2\pi - \text{periodic}\},$$

we can introduce composed Padé-type approximation to every complex-valued $2\pi -$periodic function in $L^p[-\pi,\pi]$ ($1 \le p \le \infty$) as follows.

Definition 1.3.12. *Let* $f = f_1 + i f_2$ *be a complex-valued* $2\pi -$*periodic function in* $L^p[-\pi,\pi]$. *For* $j = 1,2$, *let*

$$c_0^{(j)}, c_{\pm 1}^{(j)}, c_{\pm 2}^{(j)}, \ldots$$

be the Fourier coefficients of f_j. *Consider the two linear functionals*

$$T_{f_1} : \mathbb{P}(\mathbb{C}) \to \mathbb{C} : x^\nu \mapsto T_{f_1}\left(x^\nu\right) := c_\nu^{(1)} \text{ and } T_{f_2} : \mathbb{P}(\mathbb{C}) \to \mathbb{C} : x^\nu \mapsto T_{f_2}\left(x^\nu\right) := c_\nu^{(2)},$$

and, for $j = 1,2$, *define again*

$$V_{m+1}^{(j)}*(x) := x^{m+1} V_{m+1}^{(j)}\left(x^{-1}\right),\ V_{m+1}^{(j)}(x) := \gamma_j \prod_{k=0}^{m}\left(x - \pi_{m,k}^{(j)}\right),$$

$$W_m^{(j)}*(z) := z^m W_m^{(j)}\left(z^{-1}\right).,\ W_m^{(j)}(z) = T_{f_j}\left(\frac{V_{m+1}^{(j)}(x) - V_{m+1}^{(j)}(z)}{x - z}\right)$$

($\gamma_j \in \mathbb{C} - \{0\}$).*The complex-valued continuous function*

$$(m/m+1)_f(t) := \mathrm{Re}(m/m+1)_{f_1}(t) + i\,\mathrm{Re}(m/m+1)_{f_2}(t)$$

$$= 2\,\mathrm{Re}\,T_{f_1}\left(Q_m^{(1)}\left(x,e^{it}\right)\right) + i2\,\mathrm{Re}\,T_{f_2}\left(Q_m^{(2)}\left(x,e^{it}\right)\right) - \left[c_0^{(1)} + ic_0^{(2)}\right]$$

$$= 2\,\mathrm{Re}\frac{W_m^{(1)}*\left(e^{it}\right)}{V_{m+1}^{(1)}*\left(e^{it}\right)} + i2\,\mathrm{Re}\frac{W_m^{(2)}*\left(e^{it}\right)}{V_{m+1}^{(2)}*\left(e^{it}\right)} - \left[c_0^{(1)} + ic_0^{(2)}\right]$$

$(-\pi \le t \le \pi)$ *is called a composed Padé-type approximant to* f *. Obviously, if* $c_0^{(1)} + ic_0^{(2)} = 0$, *the composed Padé-type approximant* $(m/m+1)_f(t)$ *to* $f(t)$ *has coordinates the real parts of rational functions of type* $(m, m+1)$ *with respect to the variable* $s = s(t) = e^{it}$.

The error of such an approximation is given by a direct application of *Theorems 1.3.8* and *1.3.9* and of *Proposition 1.3.10*, via *Minkowski' s Inequality*:

Theorem 1.3.13.(a). *If* $u(z) = u_1(z) + iu_2(z) \in L^p(C), 1 \le p \le \infty$ *(respectively, if* $u(z) = u_1(z) + iu_2(z)$ *is continuous on* C *) then*

$$(m/m+1)_u\left(e^{it}\right) - u\left(e^{it}\right) = 2\lim_{r\to 1}\{\mathrm{Re}\left[\frac{1}{V_{m+1}^{(1)}\left(r^{-1}e^{it}\right)}T_{u_1}\left(\frac{V_{m+1}^{(1)}(x)}{xre^{it}-1}\right)\right]$$

$$+ i\,\mathrm{Re}\left[\frac{1}{V_{m+1}^{(2)}\left(r^{-1}e^{it}\right)}T_{u_2}\left(\frac{V_{m+1}^{(2)}(x)}{xre^{it}-1}\right)\right]\}$$

$$= \frac{1}{\pi}\lim_{r\to 1}\{\int_{-\pi}^{\pi}[u_1\left(e^{is}\right)\mathrm{Re}\left(\frac{V_{m+1}^{(1)}\left(e^{-is}\right)}{V_{m+1}^{(1)}\left(r^{-1}e^{-it}\right)}\frac{1}{re^{i(t-s)}-1}\right)$$

$$+iu_2\left(e^{is}\right)\mathrm{Re}\left(\frac{V_{m+1}^{(2)}\left(e^{-is}\right)}{V_{m+1}^{(2)}\left(r^{-1}e^{-it}\right)}\frac{1}{re^{i(t-s)}-1}\right)\,]\,ds\,\}\,,$$

in $L^p[-\pi,\pi]$ *(respectively, uniformly on* $[-\pi,\pi]$*).*

(b). *If* $f(t)=f_1(t)+i\,f_2(t)\in L^p[-\pi,\pi]$, $1\le p\le\infty$, *is a* $2\pi-$*periodic complex-valued function (respectively, if* $f(t)=f_1(t)+i\,f_2(t)$ *is a* $2\pi-$*periodic complex-valued function, continuous on* $[-\pi,\pi]$*), then there holds*

$$(m/m+1)_f(t)-f(t)=2\lim_{r\to1}\{\mathrm{Re}\left[\frac{1}{V_{m+1}^{(1)}\left(r^{-1}e^{-it}\right)}T_{f_1}\left(\frac{V_{m+1}^{(1)}(x)}{xre^{it}-1}\right)\right]$$

$$+i\,\mathrm{Re}\left[\frac{1}{V_{m+1}^{(2)}\left(r^{-1}e^{-it}\right)}T_{f_2}\left(\frac{V_{m+1}^{(2)}(x)}{xre^{it}-1}\right)\right]\}$$

$$=\frac{1}{\pi}\lim_{r\to1}\{\int_{-\pi}^{\pi}[f_1(s)\,\mathrm{Re}\left(\frac{V_{m+1}^{(1)}\left(e^{-is}\right)}{V_{m+1}^{(1)}\left(r^{-1}e^{-it}\right)}\frac{1}{re^{i(t-s)}-1}\right)$$

$$+if_2(s)\,\mathrm{Re}\left(\frac{V_{m+1}^{(2)}\left(e^{-is}\right)}{V_{m+1}^{(2)}\left(r^{-1}e^{-it}\right)}\frac{1}{re^{i(t-s)}-1}\right)]\,d\,s\,\}$$

in $L^p[-\pi,\pi]$ *(respectively, uniformly on* $[-\pi,\pi]$*).*

The property justifying the notation composed Padé-type approximant is proved in the following *Theorem.*

Theorem 1.3.14.(a). *Given any complex-valued function* $u(z) \in L^p(C)$, $1 \le p \le \infty$, *the Fourier series expansion of* $(m/m+1)_u$ *matches the Fourier series expansion of* u *up to the* $\pm m^{th}$ *Fourier term.*

(b). *Given any complex-valued* $2\pi -$ *periodic function* $f(t) \in L^p[-\pi,\pi]$, $1 \le p \le \infty$, *the Fourier series representation of* $(m/m+1)_f$ *matches the Fourier series representation of* f *up to the* $\pm m^{th}$ *Fourier term.*

Proof. Suppose that

$$\sum_{\nu=-\infty}^{\infty} \sigma_\nu e^{i\nu t} \text{ and } \sum_{\nu=-\infty}^{\infty} d_\nu^{(m)} e^{i\nu t}$$

are the Fourier series expansions of $u(e^{it})$ and $(m/m+1)_u(e^{it})$, respectively. Define the Poisson integrals of $u(e^{it})$ and $(m/m+1)_u(e^{it})$ by setting

$$u(re^{it}) = \sum_{\nu=-\infty}^{\infty} \sigma_\nu r^{|\nu|} e^{i\nu t} \quad \text{and} \quad (m/m+1)_u(re^{it}) = \sum_{\nu=-\infty}^{\infty} d_\nu^{(m)} r^{|\nu|} e^{i\nu t},$$

respectively ($0 \le r < 1, -\pi \le t \le \pi$). By *Theorem 1.3.7*, it holds $\sigma_\nu = d_\nu^{(m)}$ for any $\nu = 0, \pm 1, \pm 2, \ldots, m$. This proves *Part (a)*. Repetition of this *Proof* with only formal changes to substitute $u(e^{it})$ with $f(t)$ completes the *Proof* of the *Theorem*.

Remark 1.3.15. From *Theorem 1.3.14*, it follows that computing a composed Padé-type approximant $(m/m+1)_u(z)$ to a complex-valued function $u(z) \in L^p(C)$, (or to a complex-valued $2\pi -$ periodic function $f(t) \in L^p[-\pi,\pi]$ requires only the knowledge of

$$\sigma_0^{(j)}, \sigma_{\pm 1}^{(j)}, \sigma_{\pm 2}^{(j)}, \ldots, \sigma_{\pm m}^{(j)}, \quad j = 1,2$$

(respectively, the knowledge of

$$c_0^{(j)}, c_{\pm 1}^{(j)}, c_{\pm 2}^{(j)}, \ldots, c_{\pm m}^{(j)}, \quad j = 1,2).$$

Remark 1.3.16. We can generalize the definition of a composed Padé-type approximation as follows: If we have to approximate the complex-valued L^p-(or simply, continuous) functions

$$u(z)=u^{(1)}(z)+iu^{(2)}(z)=\sum_{\nu=-\infty}^{\infty}\left(\sigma_\nu^{(1)}+i\sigma_\nu^{(2)}\right)z^\nu \quad (|z|=1)$$

and

$$f(t)=f^{(1)}(t)+if^{(2)}(t)=\sum_{\nu=-\infty}^{\infty}\left(c_\nu^{(1)}+ic_\nu^{(2)}\right)e^{i\nu t} \quad (-\pi\le t\le\pi)$$

(with $f^{(1)}$ and $f^{(2)}$ $2\pi-$periodic, i.e. satisfying $f^{(1)}(-\pi)=f^{(1)}(\pi)$ and $f^{(2)}(-\pi)=f^{(2)}(\pi)$), then the functions

$C\to\mathbb{C}$:

$$z\mapsto 2\,\mathrm{Re}\left(\sum_{\nu=0}^{n_1-1}\sigma_\nu^{(1)}z^\nu+z^{n_1}T_{u_{n_1}^{(1)}}(Q_m(x,z))\right)+i2\,\mathrm{Re}\left(\sum_{\nu=0}^{n_2-1}\sigma_\nu^{(2)}z^\nu+z^{n_2}T_{u_{n_2}^{(2)}}(Q_m(x,z))\right) -\left[\sigma_0^{(1)}+i\sigma_0^{(2)}\right]$$

and

$C\to\mathbb{C}$:

$$z\mapsto 2\,\mathrm{Re}\left(\sum_{\nu=0}^{n_1-1}\sigma_\nu^{(1)}z^\nu+z^{n_1}T_{u_{n_1}^{(1)}}(Q_m(x,z))\right)+i2\,\mathrm{Re}\left(\sum_{\nu=0}^{n_2-1}\sigma_\nu^{(2)}z^\nu+z^{n_2}T_{u_{n_2}^{(2)}}(R_m(x,z))\right) -\left[\sigma_0^{(1)}+i\sigma_0^{(2)}\right]$$

($n_j\ge 1$) are the composed Padé-type approximants to $u(z)$. The functionals

$$T_{u_{n_1}^{(1)}}:\mathbb{P}(\mathbb{C})\to\mathbb{C} \quad\text{and}\quad T_{u_{n_2}^{(2)}}:\mathbb{P}(\mathbb{C})\to\mathbb{C},$$

associated with the series

$$u_{n_1}^{(1)}(z)=\sum_{\nu=0}^{\infty}\sigma_{n_1+\nu}^{(1)}z^{\nu} \text{ and } u_{n_2}^{(2)}(z)=\sum_{\nu=0}^{\infty}\sigma_{n_2+\nu}^{(2)}z^{\nu}$$

respectively, are defined as above. Further, $Q_m(x,z)$ and $R_m(x,z)$ denote any two polynomials, with degrees at most $m+1$, interpolating $(1-xz)^{-1}$ at x ($|x|\leq 1$). Similarly, the functions

$[-\pi,\pi]\to\mathbb{C}$:

$$t\mapsto 2\,\mathrm{Re}\left(\sum_{\nu=0}^{n_1-1}c_{\nu}^{(1)}e^{i\nu t}+e^{in_1t}T_{f_{n_1}^{(1)}}\left(Q_m(x,e^{it})\right)\right)+i2\,\mathrm{Re}\left(\sum_{\nu=0}^{n_2-1}c_{\nu}^{(2)}e^{i\nu t}+e^{in_2t}T_{f_{n_2}^{(2)}}\left(Q_m(x,e^{it})\right)\right)-\left[c_0^{(1)}+ic_0^{(2)}\right]$$

and

$[-\pi,\pi]\to\mathbb{C}$:

$$t\mapsto 2\,\mathrm{Re}\left(\sum_{\nu=0}^{n_1-1}c_{\nu}^{(1)}e^{i\nu t}+e^{in_1t}T_{f_{n_1}^{(1)}}\left(Q_m(x,e^{it})\right)\right)+i2\,\mathrm{Re}\left(\sum_{\nu=0}^{n_2-1}c_{\nu}^{(2)}e^{i\nu t}+e^{in_2t}T_{f_{n_2}^{(2)}}\left(R_m(x,e^{it})\right)\right)-\left[c_0^{(1)}+ic_0^{(2)}\right]$$

($n_j\geq 1$) are the composed Padé-type approximants to $f(t)$. The functionals

$$T_{f_{n_1}^{(1)}}:\mathbb{P}(\mathbb{C})\to\mathbb{C} \text{ and } T_{f_{n_2}^{(2)}}:\mathbb{P}(\mathbb{C})\to\mathbb{C}$$

associated with the series

$$f_{n_1}^{(1)}(t)=\sum_{\nu=0}^{\infty}c_{n_1+\nu}^{(1)}e^{i\nu t} \text{ and } f_{n_2}^{(2)}(t)=\sum_{\nu=0}^{\infty}c_{n_2+\nu}^{(2)}e^{i\nu t}$$

respectively, are defined as above, while $Q_m(x,z)$ and $R_m(x,z)$ always denote any two interpolation polynomials of $(1-xz)^{-1}$, with degrees at most $m+1$ ($|x|<1$).

1.3.3. Access to the Convergence Theory

In this *Paragraph*, we study the convergence of a sequence of Padé-type approximants to a real-valued function. The corresponding problem for the composed approximation case can be directly apportioned to a coordinate question.

Theorem 1.3.17. *Suppose there is a constant $K > 0$ and an open neighborhood U of the unit circle C into which the generating polynomials $V_{m+1}(x)$ fulfill*

$$K < \left| V_{m+1}(z) \right|, \textit{for any } z \in U \textit{ and any } m \textit{ sufficiently large.}$$

If the family

$$\left\{ V_{m+1}\left(e^{is}\right) : m = 0,1,2,\ldots \right\}$$

is an orthonormal bounded system in $L^2[-\pi,\pi]$, then

(a). *for any real-valued function $u \in L^1(C)$, the corresponding Padé-type approximation sequence $\left\{ \mathrm{Re}(m/m+1)_u(z) : m = 0,1,2,\ldots \right\}$ converges to $u(z)$ almost everywhere in C, that is*

$$\lim_{m\to\infty} \mathrm{Re}(m/m+1)_u(z) = u(z), \quad \text{for almost all } z \in C \ ;$$

(b). *for any real-valued $2\pi-$periodic function $f \in L^p[-\pi,\pi]$, the corresponding Padé-type approximation sequence $\left\{ \mathrm{Re}(m/m+1)_f(t) : m = 0,1,2,\ldots \right\}$ converges to $f(t)$ almost everywhere in $[-\pi,\pi]$, that is*

$$\lim_{m\to\infty} \mathrm{Re}(m/m+1)_f(t) = f(t), \quad \textit{for almost all } t \in [-\pi,\pi] \ .$$

Proof. Let u be a real-valued function in $L^1(C)$. Let also $\varepsilon > 0$ and let $\{0 < r_n < 1 : n = 0,1,2,...\}$ be a strictly increasing sequence satisfying

$$\lim_{n\to\infty} r_n = 1 \quad \text{and} \quad (r_n e^{it}) \in U, \text{ for any } n \geq 0 \text{ and } t \in [-\pi,\pi].$$

Fix any n, and note that the function

$$[-\pi,\pi] \to \mathbb{C} : s \mapsto \frac{u(e^{is})}{r_n e^{i(t-s)} - 1}$$

is in $L^1[-\pi,\pi]$. By *Mercer's Theorem*, the Fourier coefficients of this function with respect to the orthonormal family $\{\overline{V_{m+1}(e^{-is})} : m = 0,1,2,...\}$ tend to zero, i.e.

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} \frac{u(e^{is})}{r_n e^{i(t-s)} - 1} V_{m+1}(e^{-is}) ds = 0.$$

This means that there exists a $M = M(\varepsilon)$ such that

$$2\left|\frac{1}{V_{m+1}(r_n^{-1}e^{it})}\right|\left|\int_{-\pi}^{\pi} \frac{u(e^{is})}{r_n e^{i(t-s)} - 1} V_{m+1}(e^{-is}) ds\right| < \frac{\varepsilon}{2}$$

for any $m \geq M$ and any $n \geq 0$.

Now, by *Theorem 1.3.8*, there is a subsequence $\{r_{n_j} : j = 0,1,2,...\}$ of the sequence $\{r_n : n = 0,1,2,...\}$ such that

$$\mathrm{Re}(m/m+1)_u(e^{it}) - u(e^{it}) = \lim_{j\to\infty} 2\,\mathrm{Re}\left[\frac{1}{V_{m+1}(r_{n_j}^{-1}e^{-it})} T_u\left(\frac{V_{m+1}(x)}{x r_{n_j} e^{it} - 1}\right)\right]$$

for almost all $t \in [-\pi, \pi]$. Denote by $\mathbb{D}$ the set of all points t in $[-\pi, \pi]$ with this property, i.e.

$$\mathbb{D} = \left\{ t \in [-\pi, \pi] : \operatorname{Re}(m/m+1)_u\left(e^{it}\right) = \lim_{j\to\infty} 2\operatorname{Re}\left[\frac{1}{V_{m+1}\left(r_{n_j}^{-1} e^{-it}\right)} T_u\left(\frac{V_{m+1}(x)}{x r_{n_j} e^{it} - 1} \right) \right] \right\}.$$

Suppose $t \in \mathbb{D}$ and chose $m \geq M$. Then one can find a $J = J(\varepsilon, m)$ such that

$$\left| \operatorname{Re}(m/m+1)_u\left(e^{it}\right) - u\left(e^{it}\right) \right| \leq 2 \left| \frac{1}{V_{m+1}\left(r_{n_j}^{-1} e^{-it}\right)} \right| \left| T_u\left(\frac{V_{m+1}(x)}{x r_{n_j} e^{it} - 1} \right) \right| + \frac{\varepsilon}{2}$$

for any $n_j \geq n_J$. By *Proposition 1.3.10*, we get

$$\left| \operatorname{Re}(m/m+1)_u\left(e^{it}\right) - u\left(e^{it}\right) \right| \leq 2 \left| \frac{1}{V_{m+1}\left(r_{n_j}^{-1} e^{-it}\right)} \right| \left| \int_{-\pi}^{\pi} \frac{u\left(e^{is}\right)}{r_{n_j} e^{i(t-s)} - 1} V_{m+1}\left(e^{-is}\right) ds \right| + \frac{\varepsilon}{2}$$

for any $n_j \geq n_J$. So, it follows that

$$\left| \operatorname{Re}(m/m+1)_u\left(e^{it}\right) - u\left(e^{it}\right) \right| < \varepsilon$$

for any $m \geq M$ which implies that

$$\lim_{m\to\infty} \operatorname{Re}(m/m+1)_u(z) = u(z)$$

almost everywhere in C. The *Proof* of *Part (a)* is thus complete. Repetition of this *Proof* with only formal changes to substitute u, $u\left(e^{it}\right)$ and $u\left(e^{is}\right)$ by $f, f(t)$ and $f(s)$ respectively shows *(b)*.

This *Theorem* calls for some comments:

Remarks 1.3.18.(a). Since the generating polynomials $V_{m+1}(x)$ are defined by

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} \left(x - \pi_{m,k}\right), \quad \gamma \in \mathbb{C} - \{0\},$$

it is easily seen that if there is a constant $c < 1$ satisfying

$$\left|\pi_{m,k}\right| \leq c \quad \text{for any } m \text{ and } k,$$

then there exists an open neighborhood U of the unit circle into which there holds

$$0 < K \leq \inf_{z \in \overline{U}} \left|V_{m+1}(z)\right| \quad (m = 0,1,2,\ldots),$$

for some positive constant K which is independent of m.

(b). If the generating polynomial

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} \left(x - \pi_{m,k}\right)$$

is expressed in the form

$$V_{m+1}(x) = \sum_{k=0}^{m+1} b_k^{(m)} x^k,$$

then the orthogonality assumption for the family $\left\{V_{m+1}\left(e^{is}\right) : m = 0,1,2,\ldots\right\}$ is completely described by the following two conditions:

$$\sum_{k=0}^{m+1} \left|b_k^{(m)}\right|^2 = \frac{1}{2\pi}, \text{ for any } m$$

and

$$\sum_{k=0}^{m+1} b_k^{(m)} \overline{b_k^{(n)}} = 0 \text{, for } m < n .$$

Indeed, for any m, we have

$$2\pi \sum_{k=0}^{m+1} \left| b_k^{(m)} \right|^2 = \sum_{k,\nu=0}^{m+1} b_k^{(m)} \overline{b_\nu^{(m)}} \int_{-\pi}^{\pi} e^{iks} e^{-i\nu s} ds$$

$$= \int_{-\pi}^{\pi} V_{m+1}\left(e^{is}\right) \overline{V_{m+1}\left(e^{is}\right)} ds$$

$$= \int_{-\pi}^{\pi} \left| V_{m+1}\left(e^{is}\right) \right|^2 ds = 1,$$

and if $m < n$

$$2\pi \sum_{k=0}^{m+1} b_k^{(m)} \overline{b_k^{(n)}} = \sum_{k=0}^{m+1} \sum_{\nu=0}^{n+1} b_k^{(m)} \overline{b_\nu^{(n)}} \int_{-\pi}^{\pi} e^{iks} e^{-i\nu s} ds$$

$$= \int_{-\pi}^{\pi} V_{m+1}\left(e^{is}\right) \overline{V_{n+1}\left(e^{is}\right)} ds = 0.$$

(c). If we write

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} \left(x - \pi_{m,k}\right) = \sum_{k=0}^{m+1} b_k^{(m)} x^k ,$$

then the boundedness property for the family

$$\left\{ V_{m+1}\left(e^{is}\right) : m = 0,1,2,\ldots \right\}$$

results from the existence of a positive constant $\sigma < \infty$ satisfying

$$\sum_{k=0}^{m+1} \left| b_k^{(m)} \right| < \sigma, \quad \text{for any } m .$$

After these remarks, *Theorem 1.3.17* can be rephrased as follows:

Corollary 1.3.19. *Let the generating polynomials of a Padé-type approximation*

$$V_{m+1}(x)=\sum_{k=0}^{m+1} b_k^{(m)}x^k=\gamma\prod_{k=0}^{m}\left(x-\pi_{m,k}\right)\ \ (m=0,1,2,\ldots)$$

be chosen so that

$$\sum_{k=0}^{m+1}\left|b_k^{(m)}\right|^2=\frac{1}{2\pi}\ \ (m\geq 0)\ \text{ and }\ \sum_{k=0}^{m+1} b_k^{(m)}\overline{b_k^{(n)}}=0\ \ (m<n).$$

If there are two constants $\sigma<\infty$ *and* $c<1$ *fulfilling*

$$\sum_{k=0}^{m+1}\left|b_k^{(m)}\right|<\sigma\ \ (m\geq 0)\ \text{ and }\ \left|\pi_{m,k}\right|<c\ \ (m\geq 0,\ 0\leq k\leq m),$$

then

(a). *for any real-valued function* $u\in L^1(C)$ *the corresponding Padé-type approximation sequence* $\{\mathrm{Re}(m/m+1)_u(z): m=0,1,2,\ldots\}$ *converges to* $u(z)$ *almost everwhere on* C ;

(b). *for any real-valued* $2\pi-$*periodic function* $f\in L^1[-\pi,\pi]$ *the corresponding Padé-type approximation sequence* $\{\mathrm{Re}(m/m+1)_f(z): m=0,1,2,\ldots\}$ *converges to* $f(t)$ *almost everywhere on* $[-\pi,\pi]$.

From *Egorov's Theorem*, it now follows a uniform convergence theoretical result:

Corollary 1.3.20. *Under the assumptions of Corollary 1.3.19,*

(a). *for any* $\varepsilon > 0$ *and any real-valued function* $u \in L^1(C)$, *there is a measurable set* $\mathrm{L} \subset C$ *of Lebesgue measure* $d\lambda(\mathrm{L}) < \varepsilon$ *such that the corresponding Padé-type approximation sequence converges to* u *uniformly on* $C - \mathrm{L}$;

(b). *for any* $\varepsilon > 0$ *and any real-valued* $2\pi -$*periodic function* $f \in L^1[-\pi,\pi]$ ($f(-\pi) = f(\pi)$), *there is a measurable set* $\mathrm{E} \subset [-\pi,\pi]$ *of Lebesgue measure* $d\lambda(\mathrm{E}) < \varepsilon$ *such that the corresponding Padé -type approximation converges to* f *uniformly on* $[-\pi,\pi] - \mathrm{E}$.

All the above convergence results hold almost everywhere. However, in view of the *Proposition 1.3.10*, the *Proofs* of *Theorem 1.3.17* and of its *Corollary 1.3.19* can be directly extended to obtain more concrete results, whenever u or f is continuous:

Theorem 1.3.21. *Suppose there is a constant* $K > 0$ *and an open neighborhood* U *of the unit circle into which the generating polynomials* $V_{m+1}(x)$ *satisfy*

$$K \leq |V_{m+1}(z)|, \text{ for any } z \in U \text{ and any } m \text{ sufficiently large.}$$

If the family

$$\{V_{m+1}(e^{is}) : m = 0,1,2,...\}$$

is orthonormal in $L^2[-\pi,\pi]$, *then*

(a). *for any real-valued continuous function* u *on* C, *the corresponding sequence* $\{\mathrm{Re}(m/m+1)_u(z): m=0,1,2,...\}$ *of Padé-type approximants to* $u(z)$ *converges to* $u(z)$ *everywhere on* C, i.e.

$$\lim_{m\to\infty}\mathrm{Re}(m/m+1)_u(z)=u(z), \qquad \textit{for any } z\in C \;;$$

(b). *for any real-valued* $2\pi-$*periodic continuous function* f *on* $[-\pi,\pi]$, *the corresponding sequence* $\{\mathrm{Re}(m/m+1)_f(t): m=0,1,2,...\}$ *of Padé-type approximants to* $f(t)$ *converges to* $f(t)$ *everywhere on* $[-\pi,\pi]$, i.e.

$$\lim_{m\to\infty}\mathrm{Re}(m/m+1)_f(t)=f(t), \qquad \textit{for any } t\in[-\pi,\pi]\;.$$

Corollary 1.3.22. *Let the generating polynomials of a Padé-type approximation*

$$V_{m+1}(x)=\sum_{k=0}^{m+1}b_k^{(m)}x^k=\gamma\prod_{k=0}^{m}\left(x-\pi_{m,k}\right)$$

be such that

$$\sum_{k=0}^{m+1}\left|b_k^{(m)}\right|^2=\frac{1}{2\pi}\quad(m\geq 0)\quad \textit{and}\quad \sum_{k=0}^{m+1}b_k^{(m)}\overline{b_k^{(n)}}=0\quad(n>m).$$

If there are two constants $\sigma<\infty$ *and* $c<1$ *fulfilling*

$$\sum_{k=0}^{m+1}\left|b_k^{(m)}\right|<\sigma\quad(m\geq 0)\quad\text{and}\quad\left|\pi_{m,k}\right|<c\quad(m\geq 0, 0\leq k\leq m),$$

then

(a). *for any real-valued continuous function* u *of* C, *the corresponding Padé-type approximation sequence* $\{\mathrm{Re}(m/m+1)_u(z): m=0,1,2,...\}$ *converges to* $u(z)$ *everywhere on* C;

(b). *For any-real valued continuous* $2\pi-$*periodic function* f *in* $[-\pi,\pi]$, *the corresponding Padé-type approximation sequence* $\{\mathrm{Re}(m/m+1)_f(t): m=0,1,2,...\}$ *converges to* $f(t)$ *everywhere on* $[-\pi,\pi]$.

In [45], we gave a stronger sufficient convergence condition in terms of the entries $\pi_{m,k}$ only: If the interpolation points $\pi_{m,k}$ ($m \geq 0, 0 \leq k \leq m$) are chosen so that

$$-1<\pi_{m,k}<1 \quad \text{and} \quad \lim_{m\to\infty} \sum_{n\geq 1}\frac{1}{n}\sum_{k=0}^{m}\left(\pi_{m,k}\right)^n=-\infty,$$

then, for any real-valued $2\pi-$periodic continuous function f defined on $[-\pi,\pi]$, there holds

$$\lim_{m\to\infty} \mathrm{Re}(m/m+1)_f(t)=f(t) \qquad (-\pi\leq t\leq\pi).$$

Another reasonable approach to the convergence problem of Padé-type approximants to continuous functions can be adopted in analogy with previous results of *Paragraph 1.2.2*.

Without loss of generality, we may assume that the interpolation is taken in the Hermite sense, i.e.

$$G_m(x,z)=R_m(x,z)=(1-xz)^{-1}\left(1-\frac{V_{m+1}(x)}{V_{m+1}\left(z^{-1}\right)}\right) \quad ([23]).$$

Let u be again a real-valued continuous function on the unit circle C. Then the Poisson integral $u(z)=u\left(re^{it}\right)$ is harmonic and real-valued into the unit disk D. Suppose the generating polynomials

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} \left(x - \pi_{m,k}\right)$$

($\gamma \in \mathbb{C} - \{0\}$) of a Padé-type approximation satisfy

$$\lim_{m\to\infty} \frac{V_{m+1}(x)}{V_{m+1}\left(z^{-1}\right)} = 0$$

compactly in an open subset ω of $\mathbb{C}^2$ containing $\left(D \times \overline{D}\right) \cup (\mathbb{C} \times \{0\})$. By *Theorem 1.2.12*, the corresponding sequence $\{\mathrm{Re}(m/m+1)_u(z) : m = 0,1,2,...\}$ of Padé-type approximants converges to $u(z)$ compactly in

$$g(\omega) = \{z \in D : (\zeta, z) \in \omega, \textit{ whenever } |\zeta| \le 1\}.$$

As $D \times \overline{D} \subset \omega$, we see that $g(\omega) = D$. This means that

$$\lim_{m\to\infty} \mathrm{Re}(m/m+1)_u(z) = u(z),$$

compactly in D. Since

$$\lim_{r\to 1} \mathrm{Re}(m/m+1)_u\left(r e^{it}\right) = \mathrm{Re}(m/m+1)_u\left(e^{it}\right) \quad \text{and} \quad \lim_{r\to 1} u\left(r e^{it}\right) = u\left(e^{it}\right)$$

uniformly on $[-\pi, \pi]$, we conclude that

$$\lim_{m\to\infty} \mathrm{Re}(m/m+1)_u(z) = u(z), \quad \text{for any } z \in C .$$

Via the identification of the space of all real-valued continuous functions on C with the space of all real-valued $2\pi-$periodic continuous functions on $[-\pi, \pi]$, we have thus proved the

Theorem 1.3.23. *If the generating polynomials*

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} \left(x - \pi_{m,k}\right)$$

of a Padé-type approximation satisfy

$$\lim_{m \to \infty} \frac{V_{m+1}(x)}{V_{m+1}\left(z^{-1}\right)} = 0$$

compactly in an open set $\omega \subset \mathbb{C}2$ containing $\left(D \times \overline{D}\right) \cup (\mathbb{C} \times \{0\})$, *then*

(a). *for any real-valued continuous function* u *of* $\mathbb{C}$, *the corresponding Padé-type approximation sequence* $\{\mathrm{Re}(m/m+1)_u(z) : m = 0,1,2,...\}$ *converges to* $u(z)$ *everywhere on* C;

(b). *for any real-valued continuous* $2\pi -$*periodic function* f *in* $[-\pi, \pi]$, *the corresponding Padé-type approximation sequence* $\{\mathrm{Re}(m/m+1)_f(t) : m = 0,1,2,...\}$ *converges to* $f(t)$ *everywhere on* $[-\pi, \pi]$.

Combination of *Theorem 1.3.23* with *Corollary 1.2.14.(a)* leads to the following concrete and trivial example:

Corollary 1.3.24. *If the generating polynomials* $V_{m+1}(x)$ *have the form*

$$V_{m+1}(x) = x^{m+1} \quad (m = 0,1,2,...),$$

then the corresponding sequence $\{\mathrm{Re}(m/m+1)_u(z) : m = 0,1,2,...\}$ *of Padé-type approximants to any real-valued continuous function* $u(z)$ *on* C *converges to* $u(z)$ *everywhere on* C. *Similarly, if the generating polynomials* $V_{m+1}(x)$ *have the form*

$$V_{m+1}(x) = x^{m+1} \quad (m = 0,1,2,\ldots),$$

then the corresponding sequence $\{\mathrm{Re}(m/m+1)_f(t) : m = 0,1,2,\ldots\}$ *of Padé-type approximants to any real-valued* $2\pi-$*periodic continuous function* f *on* $[-\pi,\pi]$ *converges to* $f(t)$ *everywhere on* $[-\pi,\pi]$.

More generally, we quote the following result, whose proof is similar to that of *Theorem 1.3.23*.

Theorem 1.3.25. *If the generating polynomials* $V_{m+1}(x)$ *of a Padé-type approximation satisfy*

$$\lim_{m\to\infty} \frac{V_{m+1}(x)}{V_{m+1}(z^{-1})} = 0$$

compactly in an open subset ω *of* $\mathbb{C}^2$ *containing* $(\mathbb{C}\times\{0\}) \cup (\overline{D}\times U)$, *where* $U = U(\theta_1,\theta_2)$ *is a sector in the open disk of the form* $U = \{ re^{it} : 0 \le r < 1, -\pi \le \theta_1 \le t \le \theta_2 \le \pi \}$, *then*

(a). *for every real-valued continuous function* u *on* C, *the corresponding Padé-type approximation sequence* $\{\{\mathrm{Re}(m/m+1)_u(z) : m = 0,1,2,\ldots\}$ *converges to* $u(z)$ *for any* $z \in C \cap \overline{U}$;

(b). *for every real-valued* $2\pi-$*periodic continuous function* f *on* $[-\pi,\pi]$, *the corresponding Padé-type approximation sequence* $\{\mathrm{Re}(m/m+1)_f(t) : m = 0,1,2,\ldots\}$ *converges to* $f(t)$ *for any* $t \in [\theta_1,\theta_2]$.

Let us finally see how the problem of the convergence for a sequence of Padé-type approximants is connected with Schur and Szegö's theories.

As it is mentioned in *Theorems 1.3.17* and *1.3.21*, the crucial sufficient condition for such a convergence is the orthonormality of the system

$$\{V_{m+1}(e^{is}) : m = 0,1,2,...\}$$

into $L^2[-\pi,\pi]$ (where $\{V_{m+1}(x) : m = 0,1,2,...\}$ is the sequence of generating polynomials for this approximation).

Remind that, more generally, if $f(s)$ is a nonnegative $2\pi-$periodic measurable real-valued function on the real line with

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)ds = 1,$$

then a unique system of polynomials $\{V_{m+1}(x) : m = 0,1,2,...\}$ exists such that

- for all $m \geq 0$, $V_{m+1}(x)$ has precise degree $(m+1)$ and the coefficient of x^{m+1} is real and positive, and
- for all $m \geq 0, n \geq 0$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} V_{m+1}(e^{is})\overline{V_{n+1}(e^{is})}f(s)ds = \delta_{m+1,n+1},$$

where $\delta_{m+1,n+1}$ is *the Krönecker symbol*. One can then obtain some interesting recurrence results about the form of $V_{m+1}(x)$, dues to the connection between Schur and Szegö's theories. This connection is often attributed to Akhiezer [1], but it appears earlier and in greater detail in Geronimus [63] and [64]. It is based on important recurrence relations which were first given in Szegö's book [138]. Denoting by $\tilde{U}_{m+1}{}^*(x)$ the polynomials

$$x^{m+1}\overline{V_{m+1}(1/\overline{x})},$$

these relations were written by Geronimus in terms of the monic polynomials

$$V_{m+1}(x) := V_{m+1}(x)/\tilde{U}_{m+1}{}^*(0)$$

in the form

$$V_{m+2}(x) = xV_{m+1}(x) - \overline{a_{m+1}}\, \tilde{U}_{m+1}^{*}(x),\ m \geq 0$$

for certain parameters $a_{m+1} \in \mathbb{C},\ m \geq 0$. Since

$$g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{is}+x}{e^{is}-x} f(s)ds$$

has positive real part in the open unit disk and value 1 at the origin, the function

$$\Phi(x) = \frac{1}{x}\frac{g(x)-1}{g(x)+1}$$

belongs to the class $S(D)$ of all analytic functions which are bounded by one on the unit disk D. Geronimus showed that the Schur parameters for $\Phi(x)$ coincide with the numbers a_{m+1} in the recurrence formula. In current terminology, the numbers $V_{m+2}(0) = -\overline{a_m}$ are called *Szegö parameters*. Remind that, following Schur's construction, there is a one-to-one correspondence between $S(D)$ and the set of all sequences $\{\gamma_{m+1} : m = 0,1,2,...\}$ of complex numbers which are bounded by one and such that if some term has unit modulus, then all subsequent terms are zero. In fact, given any $\Phi(x) \in S(D)$, define a sequence

$$\Phi_1(x), \Phi_2(x), \Phi_3(x) \ldots \text{ in } S(D)$$

by setting $\Phi_1(x) \equiv \Phi(x)$ and

$$\Phi_{m+1}(x) = \frac{\Phi_m(x) - \Phi_m(0)}{x\left(1 - \overline{\Phi_m(0)}\right)\Phi_m(x)},\quad m \geq 1.$$

If $|\Phi_i(0)| = 1$, for some i, then $\Phi_i(x)$ is constant and we take $\Phi_j(x) = 0$ for any $j > i$. This occurs if and only if $\Phi(x)$ is a finite Blaschke product of i factors:

$$\Phi(x) = c\frac{x-b_1}{1-\overline{b_1}x}\frac{x-b_2}{1-\overline{b_2}x}\cdots\frac{x-b_i}{1-\overline{b_i}x},$$

where $b_1, b_2, \ldots, b_i$ are points in D and where $|c| = 1$. The numbers $\gamma_{m+1} = \Phi_{m+1}(0)$ ($m \geq 0$) are the Schur parameters for $\Phi(x)$. This method of labeling $S(D)$ by numerical sequences is known as *the Schur Algorithm* and is due to Schur ([126]). The *Schur Problem*, or *Carathéodory-Fejér Problem*, were to find conditions for the existence of a function in $S(D)$ whose initial Taylor coefficients are given numbers $t_0, t_1, t_2, \ldots, t_m$. In [126], Schur showed that such a function exists if and only if the lower triangular matrix

$$\begin{pmatrix} t_0 & & & \\ t_1 & t_0 & & \\ & \ddots & \ddots & \\ t_m & \cdots & t_1 & t_0 \end{pmatrix}$$

is bounded by one as an operator on complex Euclidean space, and he determined how all solutions can be found. The method was adapted to Pick-Nevanlinna interpolation by Nevanlinna in [48]. This variant of the problem asks to find a function in $S(D)$ which takes given values $w_1, w_2, \ldots, w_m$ at specified points $z_1, z_2, \ldots, z_m$ in D.

Under the same assumptions for the function $f(t)$ and the polynomials $V_{m+1}(x)$, we also obtain some interesting limit properties, when $f(t)$ has a factorization

$$f(t) = \left|F\left(e^{it}\right)\right|^2,$$

where $F(z)$ is an outer function in the Hardy class $H^2(D)$ on the unit disk D and $F(0)$ is positive. The term "*outer*" means that the functions

$$F(z), z\,F(z), z^2 F(z), \ldots,$$

span a dense subspace of the Hardy space, and, in particular, that $F(z)$ has no zeros in D. For all points x in the unit disk D, there holds

$$\lim_{m\to\infty} \tilde{U}_{m+1}{}^*(x) = \frac{1}{F(x)}.$$

The representation of $f(t)$ in terms of $F(z)$ is called a *spectral factorization* in applications.

1.4. Applications

1.4.1. Numerical Examples

In this *Paragraph*, several examples are considered, making use of Padé-type approximants to $2\pi -$ periodic real-valued $L^p -$ functions.

Example 1.4.1. The function

$$f_a(t) = e^{at} \quad (-\pi < t < \pi \text{ and } a \neq 0),$$

has Fourier series representation

$$F_a(t) = \frac{e^{a\pi} - e^{-a\pi}}{\pi}\left[\frac{1}{2a} + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}}{\nu^2 + a^2}\big(a\cos(\nu t) - \nu\sin(\nu t)\big)\right]$$

on the interval $-\pi < t < \pi$. As it is well known,

$$f_a(t) = F_a(t) \quad \text{for any } t \in (-\pi, \pi).$$

However,

$$F_a(\pm\pi) = \cosh(a\pi) \neq e^{\pm a\pi} = f_a(\pm\pi),$$

and thus we may consider the following $2\pi -$ periodic extension for f_a on $[-\pi, \pi]$:

$$\widetilde{f}_a(t) = \begin{cases} e^{at}, if \ -\pi < t < \pi \\ \cosh(a\pi), if \ t = \pm\pi \end{cases}$$

and then approximate $\widetilde{f}_a(t)$ in the Padé-type sense.) Evidently, for every $t \in (-\pi, \pi)$, there holds

$$f_a(t) = e^{at} = F_a(t) = \frac{e^{a\pi} - e^{-a\pi}}{\pi}\left[\frac{1}{2a} + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}}{\nu^2 + a^2}\left(a\cos(\nu t) - \nu\sin(\nu t)\right)\right]$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{2a\pi} + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}\left(e^{a\pi} - e^{-a\pi}\right)}{2\pi\left(\nu^2 + a^2\right)}(a + i\nu)e^{i\nu t} + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}\left(e^{a\pi} - e^{-a\pi}\right)}{2\pi\left(\nu^2 + a^2\right)}(a - i\nu)e^{-i\nu t}$$

$$= \sum_{\nu=-\infty}^{\infty}\frac{(-1)^{\nu}\left(e^{a\pi} - e^{-a\pi}\right)(a + i\nu)}{2\pi\left(\nu^2 + a^2\right)}e^{i\nu t} \quad \text{(or } \quad = \sum_{\nu=-\infty}^{\infty}\frac{(-1)^{\nu}\left(e^{a\pi} - e^{-a\pi}\right)}{2\pi(a - i\nu)}e^{i\nu t}\text{)}.$$

Define the $\mathbb{C}$-linear functional $T_{f_a} : \mathbb{P}(\mathbb{C}) \to \mathbb{C}$ associated with f_a by

$$T_{f_a}\left(x^{\nu}\right) = c_{\nu}^{(a)} := \frac{(-1)^{\nu}\left(e^{a\pi} - e^{-a\pi}\right)(a + i\nu)}{2\pi\left(\nu^2 + a^2\right)} \qquad (\nu = 0,1,2,\dots).$$

Given any matrix

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m\geq 0, 0\leq k\leq m}$$

with complex entries $\pi_{m,k} \in D$ ($\Leftrightarrow \left|\pi_{m,k}\right| < 1$), then, for any $m \geq 0$, a Padé-type approximant to $f_a(t)$ is a function

$$\mathrm{Re}(m/m+1)_{f_a}(t) = 2\,\mathrm{Re}\,T_{f_a}\left(Q_m\left(x, e^{it}\right)\right) - c_0^{(a)}$$

$$= 2\,\mathrm{Re}\left[\frac{e^{-it}T_{f_a}\left(\dfrac{V_{m+1}\left(e^{-it}\right) - V_{m+1}(x)}{e^{-it} - x}\right)}{V_{m+1}\left(e^{-it}\right)}\right] - c_0^{(a)},$$

where $Q_m\left(x, e^{it}\right)$ is the unique interpolation polynomial of $\left(1 - xe^{it}\right)^{-1}$ at

$$\left(\pi_{m,0}, e^{it}\right), \left(\pi_{m,1}, e^{it}\right), \dots, \left(\pi_{m,m}, e^{it}\right)$$

and

$$V_{m+1}(x) := \gamma\prod_{k=0}^{m}\left(x - \pi_{m,k}\right)$$

is the generating polynomial of this approximation ($\gamma \in \mathbb{C} - \{0\}$).

We will consider different cases:

(a). First, choose $m = 4$ and $\pi_{4,0} = \pi_{4,1} = \pi_{4,2} = \pi_{4,3} = \pi_{4,4} = 0$. Then

$$V_5(x) = x^5,$$

$$e^{-it}T_{f_a}\left(\frac{V_5(e^{-it}) - V_5(x)}{e^{-it} - x}\right) = \frac{e^{-it}\left(e^{a\pi} - e^{-a\pi}\right)}{2\pi}\left[\frac{a+4i}{16+a^2} - e^{-it}\frac{a+3i}{9+a^2} + e^{-2it}\frac{a+2i}{4+a^2}\right.$$

$$\left. - e^{-3it}\frac{a+i}{1+a^2} + e^{-4it}\frac{1}{a}\right],$$

$$V_5\left(e^{it}\right) = e^{5it},$$

and

$$\mathrm{Re}(4/5)_{f_a}(t) = \frac{e^{a\pi} - e^{-a\pi}}{\pi}\left[\frac{\mathrm{Re}\left\{(a+4i)e^{4it}\right\}}{16+a^2} - \frac{\mathrm{Re}\left\{(a+3i)e^{3it}\right\}}{9+a^2} + \frac{\mathrm{Re}\left\{(a+2i)e^{2it}\right\}}{4+a^2}\right.$$

$$\left. - \frac{\mathrm{Re}\left\{(a+i)e^{it}\right\}}{1+a^2} + \frac{1}{2a}\right].$$

Since

$$\mathrm{Re}\left\{(a+4i)e^{4it}\right\} = a\cos(4t) - 2\sin(4t), \quad \mathrm{Re}\left\{(a+3i)e^{3it}\right\} = a\cos(3t) - 3\sin(3t),$$

$$\mathrm{Re}\left\{(a+2i)e^{2it}\right\} = a\cos(2t) - 2\sin(2t), \quad \mathrm{Re}\left\{(a+i)e^{it}\right\} = a\cos(t) - 3\sin(t),$$

it follows that

$$\mathrm{Re}(4/5)_{f_a}(t) = \frac{e^{a\pi} - e^{-a\pi}}{\pi}\left[\frac{1}{2a} + \sum_{\nu=1}^{4}\frac{(-1)^\nu}{\nu^2 + a^2}\left(a\cos(\nu t) - \nu\sin(\nu t)\right)\right].$$

In other words, if $m=4$ and $\pi_{4,0}=\pi_{4,1}=\pi_{4,2}=\pi_{4,3}=\pi_{4,4}=0$, then the Padé-type approximant $\mathrm{Re}(4/5)_{f_a}(t)$ is nothing else than the trigonometric polynomial formed by summing exactly the first five terms in the Fourier series $F_a(t)$ of $f_a(t)$. Unfortunately, this choice is not very successful because of the failure of the corresponding approximation in some trivial (but characteristic) cases. If, for example, $t=0$, then

$$\mathrm{Re}(4/5)_{f_a}(0)=\frac{a\left(e^{a\pi}-e^{-a\pi}\right)}{\pi}\left[\frac{430-110a^2-10a^4}{\left(16+a^2\right)\left(9+a^2\right)\left(4+a^2\right)\left(1+a^2\right)}+\frac{1}{2a^2}\right],$$

And, for $a=1$, we obtain

$$\mathrm{Re}(4/5)_{f_1}(0)\approx 5.0116 \text{ (while in such a case } f_1(0)=1\text{);}$$

similarly, for $a=-1$,

$$\mathrm{Re}(4/5)_{f_1}(0)\approx 5.0116,\ \text{(while } f_{-1}(0) \text{ equals 1).}$$

Further, if $t=1$, then

$$\mathrm{Re}(4/5)_{f_a}(1)=\frac{e^{a\pi}-e^{-a\pi}}{\pi}\left[\frac{a(-0.5)+3.4641016}{16+a^2}-\frac{a(-1)}{9+a^2}+\frac{a(-0.5)-1.7320508}{4+a^2}\right.$$

$$\left.-\frac{a(0.5)-0.8660254}{1+a^2}+\frac{1}{2a}\right],$$

and, for $a=1$, we have

$$\mathrm{Re}(4/5)_{f_1}(1)\approx 7.3521621[0.1743589+0.1-0.4464101+0.1830127+0.5]=3.7566717$$

(while in such a case $f(1)=e\approx 2.7182818$).

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(b). Choose $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = 0, \pi_{3,3} = -\frac{i}{2}$. Then

$$V_4(x) = x^4 + i\frac{x^3}{2},$$

$$e^{-it}T_{f_a}\left(\frac{V_4\left(e^{-it}\right) - V_4(x)}{e^{-it} - x}\right) = \frac{e^{a\pi} - e^{-a\pi}}{2\pi}\left(\left[\frac{1}{a}\right] + \left[\frac{i}{2a} - \frac{a+i}{1+a^2}\right]e^{it} + \left[\frac{a+2i}{4+a^2} + \frac{1-ai}{2+2a^2}\right]e^{2it}\right.$$

$$\left. + \left[\frac{-a-3i}{9+a^2} + \frac{-2+ai}{8+2a^2}\right]e^{3it}\right),$$

$$V_4\left(e^{-it}\right) = e^{-4it}\left(1 + i\frac{e^{it}}{2}\right).$$

Therefore,

$$\mathrm{Re}(3/4)_{f_a}(t) = \frac{e^{a\pi} - e^{-a\pi}}{\pi}\left\{\mathrm{Re}\left(\frac{\left[\frac{1}{a}\right] + \left[\frac{i}{2a} - \frac{a+i}{1+a^2}\right]e^{it} + \left[\frac{a+2i}{4+a^2} + \frac{1-ai}{2+2a^2}\right]e^{2it}}{1 + i\frac{e^{it}}{2}}\right.\right.$$

$$\left.\left. + \frac{\left[\frac{-a-3i}{9+a^2} + \frac{-2+ai}{8+2a^2}\right]}{1 + i\frac{e^{it}}{2}}\right) - \frac{1}{2a}\right\}.$$

Let us give a more explicit form of $\mathrm{Re}(3/4)_{f_a}(t)$. Since

$$1 + i\frac{e^{it}}{2} = \left(1 - \frac{\sin t}{2}\right) + i\frac{\cos t}{2}, \qquad \left(1 - \frac{\sin t}{2}\right)^2 + \left(\frac{\cos t}{2}\right)^2 = \frac{5}{4}\sin t,$$

$$\left[\frac{i}{2a}-\frac{a+i}{1+a^2}\right]e^{it}=\left[-\frac{a}{1+a^2}\cos t-\left(\frac{1}{2a}-\frac{1}{1+a^2}\right)\sin t\right]$$
$$+i\left[-\frac{a}{1+a^2}\sin t+\left(\frac{1}{2a}-\frac{1}{1+a^2}\right)\cos t\right],$$
$$\left[\frac{a+2i}{4+a^2}+\frac{1-ai}{2+2a^2}\right]e^{2it}=\left[\left(\frac{a}{4+a^2}+\frac{1}{2+2a^2}\right)\cos 2t-\left(\frac{2}{4+a^2}-\frac{a}{2+2a^2}\right)\sin 2t\right]$$
$$+i\left[\left(\frac{a}{4+a^2}+\frac{1}{2+2a^2}\right)\sin 2t+\left(\frac{2}{4+a^2}-\frac{a}{2+2a^2}\right)\cos 2t\right],$$
$$\left[\frac{-a-3i}{9+a^2}+\frac{-2+ai}{8+2a^2}\right]e^{3it}=\left[\left(\frac{-a}{9+a^2}+\frac{-2}{8+2a^2}\right)\cos 3t-\left(\frac{a}{8+2a^2}-\frac{3}{9+a^2}\right)\sin 3t\right]$$
$$+i\left[\left(\frac{-a}{9+a^2}+\frac{-2}{8+2a^2}\right)\sin 3t+\left(\frac{a}{8+2a^2}-\frac{3}{9+a^2}\right)\cos 3t\right],$$

we have

$$\mathrm{Re}(3/4)_{f_a}(t)=\frac{4}{5-4\sin t}\frac{e^{a\pi}-e^{-a\pi}}{\pi}\left\{\left[\frac{5}{8a}-\frac{1}{2+2a^2}\right]\right.$$
$$+\left[-\frac{1}{2a}-\frac{1}{1+a^2}+\frac{a}{8+2a^2}+\frac{1}{4+4a^2}\right]\sin t+\left[\frac{-5a}{4+4a^2}+\frac{1}{4+a^2}\right]\cos t$$
$$+\left[\frac{-2}{4+a^2}+\frac{a}{2+2a^2}\right]\sin 2t+\left[\frac{9a}{32+8a^2}+\frac{1}{2+2a^2}+\frac{-3}{36+4a^2}\right]\cos 2t$$
$$+\left[\frac{-a}{8+2a^2}+\frac{3}{9+a^2}\right]\sin 3t+\left[\frac{-a}{9+a^2}+\frac{-1}{4+a^2}\right]\cos 3t$$
$$\left.+\left[\frac{-a}{18+2a^2}+\frac{-1}{8+2a^2}\right]\sin 4t\right\}.$$

In particular, for $a = 1$, there holds

$$\mathrm{Re}(3/4)_{f_1}(t) = \frac{1}{5-4\sin t}\frac{e^{\pi}-e^{-\pi}}{10\pi}\{15+9\sin t-17\cos t-8\sin 2t+14\cos 2t$$
$$+8\sin 3t-12\cos 3t-6\sin 4t\},$$

and if $t = 0$, then

$$\mathrm{Re}(3/4)_{f_1}(0) = 0 \quad (\text{while } f_1(0) = 1).$$

Further, if $t = 1$, then

$$\mathrm{Re}(3/4)_{f_1}(1) \approx 8.4068037 \quad (\text{while } f_1(1) = e \approx 2.7182).$$

As in the preceding case, these disappoint approximation results attest the failure of our choice $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = 0$ and $\pi_{3,3} = -\frac{i}{2}$.

Before I conclude this *Example*, I must clear my conscience. I have probably convinced you that any elementary choice of $\pi_{m,k}$'s leads to a problematic approximation. Let me show why that is not entirely true.

(c). Let $m = 3$. We choose the zeros of the *Tchebycheff polynomials*

$$\mathrm{TCH}_m(X) = \cos(m\ Arc\cos X)$$

divided by $\sqrt{\pi}$ as interpolation nodes, i.e. $\pi_{3,k} = \frac{1}{\sqrt{\pi}}\cos\left[\frac{2k+1}{7}\pi\right]$:

$$\pi_{3,0} = \frac{1}{\sqrt{\pi}}\cos\frac{\pi}{7}, \quad \pi_{3,1} = \frac{1}{\sqrt{\pi}}\cos\frac{3\pi}{7}, \quad \pi_{3,2} = \frac{1}{\sqrt{\pi}}\cos\frac{5\pi}{7}, \quad \pi_{3,3} = \frac{1}{\sqrt{\pi}}\cos\pi.$$

Then,

$$V_4(x) \approx x^4 + 0.282x^3 - 0.318x^2 - 0.067x + 0.012,$$

$$V_4\left(e^{-it}\right) \approx e^{-4it}\left(1 + 0.282e^{it} - 0.318e^{2it} - 0.067e^{3it} + 0.012e^{4it}\right),$$

and

$$\frac{e^{e-it}}{V_4\left(e^{-it}\right)} T_{f_a}\left(\frac{V_4\left(e^{-it}\right) - V_4(x)}{e^{-it} - x}\right) =$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{2\pi}\left\{\frac{\left[\frac{1}{a}\right] + \left[-\frac{a+1}{1+a^2} + 0.282\frac{1}{a}\right]e^{it} + \left[\frac{a+2i}{4+a^2} - 0.282\frac{a+i}{1+a^2} - 0.318\frac{1}{a}\right]e^{2it}}{1 + 0.282e^{it} - 0.318e^{2it} - 0.067e^{3it} + 0.012e^{4it}}\right.$$

$$\left. + \frac{\left[\frac{-a+3i}{9+a^2} + 0.282\frac{a+2i}{4+a^2} + 0.318\frac{a+i}{1+a^2} - 0.067\frac{1}{a}\right]e^{3it}}{1 + 0.282e^{it} - 0.318e^{2it} - 0.067e^{3it} + 0.012e^{4it}}\right\}.$$

It follows that

$$\mathrm{Re}(3/4)_{f_a}(t) = 2\,\mathrm{Re}\left[\frac{e^{-it}}{V_4\left(e^{-it}\right)} T_f\left(\frac{V_4\left(e^{-it}\right) - V_4(x)}{e^{-it} - x}\right)\right] - c_0$$

$$= 2\,\mathrm{Re}\left[\frac{e^{-it}}{V_4\left(e^{-it}\right)} T_f\left(\frac{V_4\left(e^{-it}\right) - V_4(x)}{e^{-it} - x}\right)\right] - \frac{e^{a\pi} - e^{-a\pi}}{2a\pi}$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{\pi}\left\{(1 + 0.282\cos t - 0.318\cos 2t - 0.067\cos 3t + 0.012\cos 4t)^2\right.$$

$$\left. + (0.282\sin t - 0.318\sin 2t - 0.067\sin 3t + 0.012\sin 4t)^2\right\}^{-1}$$

$$\left\{\left[\frac{1.185137}{a} - \frac{0.21363a}{1+a^2} - \frac{0.336894a}{4+a^2} + \frac{0.067a}{9+a^2}\right]\right.$$

$$+\left[\frac{1.521358}{1+a^2}-\frac{0.51188}{4+a^2}-\frac{0.99}{9+a^2}\right]\sin t$$

$$+\left[\frac{0.426456}{a}-\frac{0.839938}{1+a^2}+\frac{0.128708a}{4+a^2}+\frac{0.306a}{9+a^2}\right]\cos t$$

$$+\left[\frac{0.25594}{1+a^2}-\frac{2.135048}{4+a^2}+\frac{0.846}{9+a^2}\right]\sin 2t$$

$$+\left[-\frac{0.677604}{a}-\frac{0.128708a}{1+a^2}+\frac{1.091524a}{4+a^2}-\frac{0.2822a}{9+a^2}\right]\cos 2t$$

$$+\left[-\frac{0.33}{1+a^2}-\frac{0.564}{4+a^2}-\frac{3}{9+a^2}\right]\sin 3t$$

$$+\left[-\frac{0.130616}{a}+\frac{0.306a}{1+a^2}+\frac{0.282a}{4+a^2}-\frac{a}{9+a^2}\right]\cos 3t$$

$$+\left[\frac{0.012}{a}\right]\cos 4t\Big\}$$

$$-\frac{e^{a\pi}-e^{-a\pi}}{2a\pi} \quad .$$

In particular, for $a=1$, there holds

$$\mathrm{Re}(3/4)_{f_1}(t)$$

$$\approx 7.35211621\{(1+0.282\cos t-0.318\cos 2t-0.067\cos 3t+0.012\cos 4t)^2$$

$$+(0.282\sin t-0.318\sin 2t-0.067\sin 3t+0.012\sin 4t)^2\}^{-1}$$

$$\{[1.0176432]$$

$$+[0.559303]\sin t+[0.0628286]\cos t$$

$$+[-0.2144396]\sin 2t+[-0.5518532]\cos 2t$$

$$+[-0.5778]\sin 3t+[-0.021216]\cos 3t$$

$$+[0.012]\cos 4t\}-3.676081 \quad .$$

Thus, if $t = 0$, then

$$\mathrm{Re}(3/4)_{f_1}(0) \approx 0.9455091 \quad (\text{,while } f_1(0) = e^0 = 1);$$

if $t = 1$, then

$$\mathrm{Re}(3/4)_{f_1}(1) \approx 2.8227598 \quad (\text{,while } f_1(1) = e^1 \approx 2.7182818);$$

if $t = e$, then

$$\mathrm{Re}(3/4)_{f_1}(e) \approx 15.968062 \quad (\text{,while } f_1(e) = e^e \approx 15.154261).$$

However, if $t = \sqrt{3}$, then

$$\mathrm{Re}(3/4)_{f_1}\left(\sqrt{3}\right) \approx 7.66529587 \quad (\text{while } f_1\left(\sqrt{3}\right) = e^{\sqrt{3}} \approx 5.6522335),$$

and if $t = \dfrac{\pi}{2}$, then

$$\mathrm{Re}(3/4)_{f_1}\left(\frac{\pi}{2}\right) \approx 5.7613728 \quad (\text{while } f_1\left(\frac{\pi}{2}\right) \approx 4.810477).$$

Example 1.4.2. Let f be the real-valued function

$$f(t) = t^2 \quad (t \in \mathbb{R}).$$

As it is easily verified, the Fourier series $F(t)$ of f into $[-\pi, \pi]$ is given by

$$F(t) = \frac{\pi^2}{3} - 4\left(\cos t - \cos\left[\frac{2t}{t^2}\right] + \cos\left[\frac{3t}{t^3}\right] - \ldots\right) = \frac{\pi^3}{3} - \sum_{\substack{\nu=-\infty \\ (\nu \neq 0)}}^{\infty} \frac{2(-1)^\nu}{\nu^2} e^{i\nu t}$$

($-\pi \leq t < \pi$). Define the $\mathbb{C}$-linear functional $T_f : \mathbb{P}(\mathbb{C}) \to \mathbb{C}$ associated with f by

$$T_f\left(x^\nu\right) = c_\nu = \begin{cases} \dfrac{\pi^2}{3}, if & \nu = 0 \\ \dfrac{2(-1)^\nu}{\nu^2}, if & \nu = 1,2,3,... \end{cases}$$

For any matrix

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m \geq 0, 0 \leq k \leq m}$$

with complex entries $\pi_{m,k} \in D \quad \left(\Leftrightarrow \left|\pi_{m,k}\right| < 1\right)$, a Padé-type approximant to $f(t)$ is a function

$$\operatorname{Re}(m/m+1)_f(t) = 2\operatorname{Re}\left[\frac{e^{-it}T_f\left(\dfrac{V_{m+1}\left(e^{-it}\right) - V_{m+1}(x)}{e^{-it} - x}\right)}{V_{m+1}\left(e^{-it}\right)}\right] - c_0 .$$

As usually,

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m}\left(x - \pi_{m,k}\right)$$

is the generating polynomial of this approximation ($\gamma \in \mathbb{C} - \{0\}$).

(a). If $m = 4$ and $\pi_{4,0} = \pi_{4,1} = \pi_{4,2} = \pi_{4,3} = \pi_{4,4} = 0$, then

$V_5(x) = x^5$,

$e^{-it}T_f\left(\dfrac{V_5\left(e^{-it}\right) - V_5(x)}{e^{-it} - x}\right) = 2e^{-5it}\left(\dfrac{\pi^2}{6} - e^{it} + \dfrac{e^{2it}}{2} - \dfrac{e^{3it}}{9} + \dfrac{e^{4it}}{16}\right)$,

$V_5\left(e^{-it}\right) = e^{-5it}$

and

$$\operatorname{Re}(4/5)_f(t) = \frac{\pi^2}{3} - 4\left(\cos t - \frac{\cos 2t}{2^2} + \frac{\cos 3t}{3^2} - \frac{\cos 4t}{4^2}\right).$$

This means that $\mathrm{Re}(4/5)_f(t)$ is the trigonometric polynomial which equals the partial sum of the first five terms in the Fourier expansion $F(t)$ of $f(t)$. Indicatively, we have

t	$f(t)$	$\mathrm{Re}(4/5)_f(t)$
0	0.0000000	0.0954237
$-\frac{\pi}{2}$	2.4674011	2.5398681
$\frac{\pi}{8}$	0.1542125	0.1313749
$\frac{\sqrt{3\pi}}{4}$	1.8505508	1.9704128
1	1.0000000	0.1091051
e	7.3890559	7.077193

(b). If $m=3$ and $\pi_{3,0}=\pi_{3,1}=\pi_{3,2}=0$, $\pi_{3,3}=-1$, then

$$V_4(x)=x^4+x^3,$$

$$e^{-it}T_f\left(\frac{V_4(e^{-it})-V_4(x)}{e^{-it}-x}\right)\approx e^{-4it}\left(3.2898681+1.2898681e^{it}-1.5e^{2it}+0.2777777e^{3it}\right),$$

$$V_4(e^{-it})=e^{-4it}(1+e^{it})$$

and

$$\mathrm{Re}(3/4)_f(t) \approx \frac{4.5797361 + 3.0797362\cos t - 1.2222224\cos 2t + 0.2777777\cos 3t}{1+\cos t} - 3.2898681.$$

Thus,

t	$f(t)$	$\mathrm{Re}(3/4)_f(t)$
0	0.0000000	0.0676457
$-\pi/2$	2.4674011	2.5120904
$\pi/8$	0.1542125	0.1755879
$\sqrt{3}\pi/4$	1.850508	1.8178327
1	1.0000000	0.9153782
e	7.3890559	6.424624
$\pi/3$	1.0966227	1.3824601
$-\pi/4$	0.6168502	0.5534881

(c). If $m = 3$ and

$$\pi_{3,k} = \frac{1}{\sqrt{\pi}} \cos\left[\frac{2k+1}{7}\pi\right],$$

or simply

$$\pi_{3,0} = \cos\left[\frac{2k+1}{7}\pi\right]$$

($0 \le k \le 3$), then

$$\begin{aligned}\mathrm{Re}(3/4)_f(t) \approx{}& 2\{[1 + 0.282\cos t - 0.318\cos 2t - 0.067\cos 3t + 0.012\cos 4t]^2 \\ & + [0.282\sin t - 0.318\sin 2t - 0.067\sin 3t + 0.012\sin 4t]^2\}^{-1} \\ & \{2.9347573 + 0.9873185\cos t + 0.0469156\cos 2t \\ & + 0.1130685\cos 3t + 0.0394784\cos 4t\} \\ & - 3.2898681,\end{aligned}$$

or

$$\begin{aligned}\mathrm{Re}(3/4)_f(t) \approx{}& 2\{[1 + 0.5\cos t - \cos 2t - 0.375\cos 3t + 0.125\cos 4t]^2 \\ & + [0.5\sin t - \sin 2t - 0.375\sin 3t + 0.125\sin 4t]^2\}^{-1} \\ & \{8.3504691 + 0.4763827\cos t - 7.0232814\cos 2t \\ & - 0.4840065\cos 3t + 0.4112335\cos 4t\} \\ & - 3.2898681,\end{aligned}$$

respectively. Both these two approximants seem to be not efficient, so their use is not recommended.

Example 1.4.3. Let f be the real-valued non-negative function $f(t)=|t|$ $(t\in\mathbb{R})$. The Fourier series of f into $[-\pi,\pi]$ is

$$F(t)=\frac{\pi}{2}-\frac{4}{\pi}\left(\cos t+\frac{\cos 3t}{3^2}+\frac{\cos 5t}{5^2}+\ldots\right)=\frac{\pi}{2}+\sum_{\nu=-\infty(\nu\neq 0)}^{\infty}\frac{(-1)^{\nu}-1}{\pi\nu^2}e^{i\nu t}.$$

Define the $\mathbb{C}$-linear functional $T_f:\mathbb{P}(\mathbb{C})\to\mathbb{C}$ associated with f by

$$T_f\left(x^{\nu}\right)=c_{\nu}=\begin{cases}\dfrac{\pi}{2}, if\ \nu=0\\ \dfrac{(-1)^{\nu}-1}{\pi\nu^2}, if\ \nu=1,2,3,\ldots\end{cases}.$$

Observe that

$$c_2=c_4=c_6=\ldots=0\ \text{, while } c_0,c_1,c_3,c_5,\ldots\neq 0.$$

For any matrix

$$\mathrm{M}=\left(\pi_{m,k}\right)_{m\geq 0,0\leq k\leq m}$$

with complex entries $\pi_{m,k}\in D$, a Padé-type approximant to $f(t)$ is a function

$$\mathrm{Re}(m/m+1)_f(t)=2\,\mathrm{Re}\left[\frac{e^{-it}T_f\left(\dfrac{V_{m+1}\left(e^{-it}\right)-V_{m+1}(x)}{e^{-it}-x}\right)}{V_{m+1}\left(e^{-it}\right)}\right]-c_0\ ,$$

where

$$V_{m+1}(x)=\gamma\prod_{k=0}^{m}\left(x-\pi_{m,k}\right)$$

is the generating polynomial of this approximation ($\gamma \in \mathbb{C}-\{0\}$).

(a). If $m=3$ and $\pi_{3,0}=\pi_{3,1}=\pi_{3,2}=\pi_{3,3}=\frac{1}{4}$, then

$$V_4(x)=\frac{1}{256}\left(256x^4-256x^3+32x^2-16x+1\right),$$

$$e^{-it}T_f\left(\frac{V_4\left(e^{-it}\right)-V_4(x)}{e^{-it}-x}\right)$$

$$=\frac{e^{-4it}}{256}\left(128\pi-\left[\frac{512}{\pi}+128\pi\right]e^{it}+\left[\frac{512}{\pi}+16\pi\right]e^{2it}-\left[\frac{512}{9\pi}+\frac{64}{\pi}+8\pi\right]e^{3it}\right),$$

$$V_4\left(e^{-it}\right)=\frac{e^{-4it}}{256}\left(256-256e^{it}+32e^{2it}-16e^{3it}+e^{4it}\right),$$

and therefore

$$\mathrm{Re}(3/4)_f(t)=2\{[256-256\cos t+32\cos 2t-16\cos 3t+\cos 4t]^2$$

$$+[256\sin t+32\sin 2t-16\sin 3t+\sin 4t]^2\}^{-1}$$

$$\{254941.5-324774.49\cos t+92997.148\cos 2t$$

$$-23283.974\cos 3t+402.12385\cos 4t\}$$

$$-1.5707963.$$

Thus

t	$f(t)$	$\mathrm{Re}(3/4)_f(t)$
0	0.0000000	0.3828912
$-\frac{\pi}{4}$	0.7853981	0.9379537
$-\frac{\pi}{6}$	0.5235987	0.894674
1	1.0000000	1.0112661

(b). If $m=4$ and $\pi_{4,0}=\pi_{4,1}=\pi_{4,2}=\pi_{4,3}=0$, $\pi_{4,4}=-1$, then

$$V_5(x)=x^5+x^4,$$

$$e^{-it}T_f\left(\frac{V_5(e^{-it})-V_5(x)}{e^{-it}-x}\right)$$

$$=e^{-5it}\left([c_0]+[c_0+c_1]e^{it}+[c_1+c_2]e^{2it}+[c_2+c_1]e^{3it}+[c_3+c_4]e^{4it}\right),$$

$$V_5(e^{-it})=e^{-5it}(1+e^{it}).$$

Hence

$$\mathrm{Re}(4/5)_f(t)$$

$$=\frac{1}{1+\cos t}\{2.5049728+2.2831853\cos t-0.7073552\cos 2t-0.141471\cos 3t$$

$$-0.0707355\cos 4t\}-1.5707963\,.$$

Thus,

t	$f(t)$	$\mathrm{Re}(4/5)_f(t)$
0	0.0000000	0.3635019
$-\frac{\pi}{4}$	0.7853981	0.9423442
$-\frac{\pi}{6}$	0.5235987	0.6606627
1	1.0000000	1.2120203

(c). If $m = 4$ and $\pi_{4,0} = \pi_{4,1} = \pi_{4,2} = \pi_{4,3} = \pi_{4,4} = 0$, then

$V_5(x) = x^5$,

$$e^{-it}T_f\left(\frac{V_5(e^{-it}) - V_5(x)}{e^{-it} - x}\right) = e^{-5it}\left(-\frac{2}{9\pi}e^{3it} - \frac{2}{\pi}e^{it} + \frac{\pi}{2}\right),$$

$V_5(e^{-it}) = e^{-5it}$.

It follows that

$$\mathrm{Re}(4/5)_f(t) = 2\,\mathrm{Re}\left(\frac{\pi}{2} - \frac{2}{9\pi}e^{3it} - \frac{2}{\pi}e^{it}\right) - \frac{\pi}{2} = \frac{\pi}{2} - \frac{4}{\pi}\cos t - \frac{4}{9\pi}\cos 3t,$$

and indicatively we have

t	$f(t)$	$\mathrm{Re}(4/5)_f(t)$
0	0.0000000	0.1560857
$-\frac{\pi}{4}$	0.7853981	0.7705152
$-\frac{\pi}{6}$	0.5235987	0.4681385
1	1.0000000	1.0756475
e	2.7182818	2.673454

Remark 1.4.4. In the preceding two *Examples* (:*1.4.2* and *1.4.3*) our indicative results corresponding to the choice "$m = 3$, $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = 0$, $\pi_{3,3} = -1$ ", seem to be persuasive and painless: the fact that the interpolation point $\pi_{3,3}$ lies in the unit circle C (in other words, the fact that $|\pi_{3,3}| = 1$) does not steal in our special computations. The pathology of such a choice will be apparent in the following *Example*.

Example 1.4.5. Assume that $r \in [0,1)$. Let f_r be the real-valued function

$$f_r : \mathbb{R} \to \mathbb{R} : t \mapsto f_r(t) = \mathrm{Im}\left(\frac{1 + re^{it}}{2(1 - re^{it})}\right) = \frac{r \sin t}{1 - 2r \cos t + r^2}.$$

As it easily verified, the Fourier series expansion $F_r(t)$ of $f_r(t)$ on $[-\pi, \pi]$ is

$$F_r(t) = \sum_{\nu=1}^{\infty} r^{\nu} \sin[\nu t] = \sum_{\nu=1}^{\infty} \frac{ir^{\nu}}{2} e^{-i\nu t} + \sum_{\nu=1}^{\infty} \frac{-ir^{\nu}}{2} e^{i\nu t} = \sum_{\nu=-\infty}^{\infty} c_{\nu} e^{i\nu t},$$

with

$$c_0 = 0 \text{ and } c_{\nu} = \overline{c_{-\nu}} = \frac{-ir^{\nu}}{2} \text{ for any } \nu \neq 0.$$

Define the $\mathbb{C}$-linear functional $T_{f_r} : \mathbb{P}(\mathbb{C}) \to \mathbb{C}$ associated with f_r by

$$T_{f_r}\left(x^{\nu}\right) = c_{\nu} = \begin{cases} 0, if\ \nu = 0 \\ \dfrac{-ir^{\nu}}{2}, if\ \nu = 1,2,3,... \end{cases},$$

and suppose the complex infinite triangular matrix

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m \geq 0, 0 \leq k \leq m}$$

is given. For any $m \geq 0$, a Padé-type approximant to $f_r(t)$ is a function

$$\mathrm{Re}(m/m+1)_{f_r}(t) = 2\,\mathrm{Re}\left[\frac{e^{-it} T_{f_r}\left(\dfrac{V_{m+1}\left(e^{-it}\right) - V_{m+1}(x)}{e^{-it} - x}\right)}{V_{m+1}\left(e^{-it}\right)}\right] \quad (,-\pi \leq t \leq \pi),$$

where

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} \left(x - \pi_{m,k}\right)$$

is the generating polynomial of this approximation ($\gamma \in \mathbb{C} - \{0\}$).

(a). If $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = 0$, $\pi_{3,3} = -i$,then

$V_4(x) = x^4 + ix^3$,

$$e^{-it} T_{f_r}\left(\frac{V_4\left(e^{-it}\right) - V_4(x)}{e^{-it} - x}\right) = e^{-4it}\left([c_0] + [c_1 + ic_0]e^{it} + [c_2 + ic_1]e^{2it} + [c_3 + ic_2]e^{3it}\right),$$

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$V_4\left(e^{-it}\right)=e^{-4it}\left(1+ie^{it}\right)$.

Hence,

$$\mathrm{Re}(3/4)_{f_r}(t)$$
$$=\frac{r}{2\,(1-\sin t)}\left\{-1+2\sin t-r\cos t+r\sin 2t+\left[1-r^2\right]\cos 2t\right.$$
$$\left.+r^2\sin 3t+r\,\cos 3t+r\sin 4t\right\}.$$

Observe that $\mathrm{Re}(3/4)_{f_r}(t)$ is well defined everywhere on $[-\pi,\pi]$, with the exception of the point $t=\dfrac{\pi}{2}$. This is a consequence of the choice $\pi_{3,3}=-i$, which in particular implies

$$\left|\pi_{3,3}\right|=1\left(\Leftrightarrow \pi_{3,3}\in C\right).$$

We have

t	$f_r(t)$	$\mathrm{Re}(3/4)_{f_r}(t)$
0	0	$-0.5r^3$
$\frac{\pi}{3}$	$\frac{0.8660254r}{(1+r)^2}$	$3.7320507\,r\left(0.7320508-1.5r+0.5r^2\right)$
$-\pi$	0	$0.5r\left(1-r^2\right)$
$\frac{\pi}{4}$	$\frac{0.7071067r}{1-1.4142135r+r^2}$	$1.7071067\,r\left(0.4142135-0.4142135r+0.7071067r^2\right)$
$-\frac{\pi}{6}$	$\frac{-0.5r}{1-1.7320508r+r^2}$	$0.3333333r\left(0.5-2.5980762r-1.5r^2\right)$
$\frac{\pi}{5}$	$\frac{0.5877852r}{1-1.6180339r+r^2}$	$1.2129599\,r\left(0.4845874+2.0388417r-0.3090169r^2\right)$
$\frac{\pi}{2}$	$\frac{r}{1+r^2}$	*undefined*

and in particular

r	1/8		1/2		3/4	
t	$f_{1/8}(t)$	$\mathrm{Re}(3/4)_{f_{1/8}}(t)$	$f_{1/2}(t)$	$\mathrm{Re}(3/4)_{f_{1/2}}(t)$	$f_{3/4}(t)$	$\mathrm{Re}(3/4)_{f_{3/4}}(t$
0	0.0000000	-0.000976562	0.000000	-0.0625000	0.0000000	-0.2109375
$\frac{\pi}{3}$	0.0855333	0.2576809	0.1924499	0.1997595	0.2120878	-0.3126503
$-\pi$	0.0000000	0.0615234	0.0000000	0.1875000	0.0000000	0.1640625
$\frac{\pi}{4}$	0.1053686	0.0962008	0.651239	0.3276649	1.0567713	0.6418305
$-\frac{\pi}{6}$	-0.0782111	0.00632594	-0.6510847	-0.1540062	-1.4233557	-0.5730767
$\frac{\pi}{5}$	0.0903316	0.1062575	0.6664487	0.9589957	1.2632407	1.6737902

(b). If $m = 3$ and $\pi_{3,k} = \cos\left[\frac{2k+1}{7}\pi\right]$ (, $k = 0,1,2,3$):

$$\pi_{3,0} = 0.9009688, \quad \pi_{3,1} = 0.2225209, \quad \pi_{3,2} = -0.6234898, \quad \pi_{3,3} = -1,$$

then

$$V_4(x) \approx x^4 + 0.5x^3 - x^2 - 0.375x + 0.125,$$

$$e^{-it}T_{f_r}\left(\frac{V_4(e^{-it}) - V_4(x)}{e^{-it} - x}\right) \approx e^{-4it}\left([c_0] + [c_1 + 0.5c_0]e^{it} + [c_2 + 0.5c_1 - c_0]e^{2it}\right.$$

$$\left. + [c_3 + 0.5c_2 - c_1 - 0.375c_0]e^{3it}\right),$$

$$V_4(e^{-it}) \approx e^{-4it}\left(1 + 0.5e^{it} - e^{2it} - 0.375e^{3it} + 0.125e^{4it}\right).$$

Since

$$c_0 = 0,\ c_1 = -i0.5r,\ c_2 = -i0.5r^2,\ c_3 = -i0.5r^3\ ,$$

we obtain

$$\mathrm{Re}(3/4)_{f_r}(t) \approx (-r)\{[1+0.5\cos t - \cos 2t - 0.375\cos 3t + 0.125\cos 4t]^2$$
$$+[0.5\sin t - \sin 2t - 0.375\sin 3t + 0.125\sin 4t]^2\}^{-1}$$
$$\{[1.125r^2 - 0.3125r - 1.0625]\sin t + [-0.5r^2 - 1.125r - 1.3125]\sin 2t$$
$$+[-r^2 + 0.5r + 1.375]\sin 3t\}.$$

Indicatively, we have

t	$f_r(t)$	$\mathrm{Re}(3/4)_{f_r}(t)$
0	0	0
$\frac{\pi}{3}$	$\frac{0.8660254r}{(1+r)^2}$	$-\frac{0.8660254r(0.625r^2 - 1.4375r - 2.375)}{4.5468749}$
$-\pi$	0	0
$\frac{\pi}{4}$	$\frac{0.7071067r}{1-1.4142135r+r^2}$	$-\frac{r(0.0883883r^2 - 0.9924175r + 0.0183059)}{2.3765699}$
$-\frac{\pi}{6}$	$-\frac{0.5r}{1-1.7320508r+r^2}$	$-\frac{r(0.8705127r^2 + 0.6305285r + 0.2929083)}{1.5370791}$
$\frac{\pi}{5}$	$\frac{0.5877852r}{1-1.6180339r+r^2}$	$\frac{r(0.7653264r^2 + 0.7775396r + 0.5654351)}{2.2188258}$
$\frac{\pi}{2}$	$\frac{r}{1+r^2}$	$-\frac{r(2.125r^2 - 0.8125r - 2.4375)}{5.28125}$

In particular, there holds

r	$1/8$		$1/2$		$3/4$	
t	$f_{1/8}(t)$	$\mathrm{Re}(3/4)_{f_{1/8}}(t)$	$f_{1/2}(t)$	$\mathrm{Re}(3/4)_{f_{1/2}}(t)$	$f_{3/4}(t)$	$\mathrm{Re}(3/4)_{f_{3/4}}(t)$
0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
$\frac{\pi}{3}$	0.0855333	0.06059	0.1924499	0.2797496	0.2120878	0.4430566
$-\pi$	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
$\frac{\pi}{4}$	0.1053686	0.0621036	0.651239	1.0849208	1.0567713	2.4146128
$-\frac{\pi}{6}$	-0.078211	-0.0313358	-0.6510847	-0.2686265	-1.4233557	-0.612591
$\frac{\pi}{5}$	0.0903316	0.0380035	0.6664487	0.2581402	1.2632407	0.5337572
$\frac{\pi}{2}$	0.1230769	0.0279106	0.4000000	0.1030281	0.4800000	0.1237382

(c). If $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = \pi_{3,3} = 0$, then

$$\mathrm{Re}(3/4)_{f_r}(t) = \sum_{\nu=1}^{3} r^{\nu} \sin[\nu t],$$

and

r	1/8		1/2		3/4	
t	$f_{1/8}(t)$	$\mathrm{Re}(3/4)_{f_{1/8}}(t)$	$f_{1/2}(t)$	$\mathrm{Re}(3/4)_{f_{1/2}}(t)$	$f_{3/4}(t)$	$\mathrm{Re}(3/4)_{f_{1/4}}(t)$
0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
$\frac{\pi}{3}$	0.0855333	0.0270632	0.1924499	0.649519	0.2120878	0.7713038
$-\pi$	0.000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
$\frac{\pi}{4}$	0.1053686	0.1053943	0.651239	0.6919419	1.0567713	1.3911406
$-\frac{\pi}{6}$	-0.078211	-0.0779847	-0.651084	-0.5915063	-1.4233557	-0.9592547
$\frac{\pi}{5}$	0.0903316	0.2239326	0.6664487	0.6505387	1.2632407	1.377035
$\frac{\pi}{2}$	0.1230769	0.1230468	0.4000000	0.2500000	0.4800000	0.328125

Example 1.4.6. Let $f(t)$ be the function

$$f(t) = \begin{cases} -\dfrac{\pi}{2}, if \ -\pi < t < 0 \\ \dfrac{\pi}{2}, if \ 0 < t < \pi, \end{cases}$$

with Fourier series expansion

$$F(t) = 2\sum_{\nu=1}^{\infty} \frac{\sin[(2\nu-1)t]}{2\nu-1} = \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} \frac{i[(-1)^{\nu}-1]}{2\nu} e^{i\nu t}.$$

Define the $\mathbb{C}$-linear functional $T_f : \mathbb{P}(\mathbb{C}) \to \mathbb{C}$ by

$$T_f\left(x^{\nu}\right) = c_{\nu} := \begin{cases} \dfrac{i[(-1)^{\nu}-1]}{2\nu}, if\ \nu = 1,2,3,\ldots \\ 0, \quad if\ \nu = 0. \end{cases}$$

If

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m\geq 0, 0\leq k\leq m}$$

is a complex infinite triangular matrix, then, for any $m \geq 0$, a Padé-type approximant to $f(t)$ is a function

$$\mathrm{Re}(m/m+1)_f(t) = 2\,\mathrm{Re}\left[\frac{e^{-it} T_f\left(\dfrac{V_{m+1}(e^{-it}) - V_{m+1}(x)}{e^{-it}-x}\right)}{V_{m+1}(e^{-it})}\right] \quad (-\pi \leq t \leq \pi),$$

where

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m}\left(x - \pi_{m,k}\right)$$

is the generating polynomial of this approximation ($\gamma \in \mathbb{C} - \{0\}$).

(a). If $m = 3$ and $\pi_{3,0} = -1$, $\pi_{3,1} = -\dfrac{1}{2}$, $\pi_{3,2} = -\dfrac{1}{3}$, $\pi_{3,3} = -\dfrac{1}{4}$, then

$$V_4(x) = \frac{1}{24}\left(24x^4 + 50x^3 + 35x^2 + 10x + 1\right),$$

$$e^{-it} T_f\left(\frac{V_4(e^{-it}) - V_4(x)}{e^{-it}-x}\right) = \frac{e^{-4it}}{24}\left([24c_1]e^{it} + [50c_1]e^{2it} + [24c_3 + 35c_1]e^{3it}\right),$$

$$V_4\left(e^{-4it}\right) = \frac{e^{-4it}}{24}\left(24 + 50e^{it} + 35e^{2it} + 10e^{3it} + e^{4it}\right).$$

It follows that

$\mathrm{Re}(3/4)_f(t)$

$$= \frac{6396\sin t + 6120\sin 2t + 2016\sin 3t}{[24+50\cos t+35\cos 2t+10\cos 3t+\cos 4t]^2+[50\sin t+35\sin 2t+10\sin 3t+\sin 4t]^2}$$

and therefore

t	$f(t)$	$\mathrm{Re}(3/4)_f(t)$
$\frac{\pi}{2}$	1.5707963	2.5764705
$-\frac{\pi}{2}$	-1.5707963	-2.5764705
$\frac{\pi}{3}$	1.5707963	1.8906633
$-\frac{\pi}{3}$	-1.5707963	-1.8906633
$\frac{\pi}{4}$	1.5707963	1.3992217
$-\frac{\pi}{4}$	-1.5707963	-1.3992217
$\frac{\pi}{5}$	1.5707963	1.1066319
$-\frac{\pi}{5}$	-1.5707963	-1.1066319
$\frac{\pi}{6}$	1.5707963	0.9153747
$-\frac{\pi}{6}$	-1.5707963	-0.9153747

(b). If $m = 4$ and $\pi_{4,0} = \pi_{4,1} = \pi_{4,2} = \pi_{4,3} = 0$, $\pi_{4,4} = -1$, then

$V_5(x) = x^5 + x^4$,

$$e^{-it}T_f\left(\frac{V_5(e^{-it})-V_5(x)}{e^{-it}-x}\right)$$
$$=e^{-it}T_f\left(e^{-4it}+e^{-3it}x+e^{-2it}x^2+e^{-it}x^3+x^4+e^{-3it}+e^{-2it}x+e^{-it}x^2+x^3\right)$$
$$=e^{-5it}\left(c_1e^{it}+c_1e^{2it}+c_3e^{3it}+c_3e^{4it}\right),$$
$$V_5(e^{-it})=e^{-5it}\left(1+e^{it}\right).$$

If follows that

$$\mathrm{Re}(4/5)_f(t)=\frac{6\sin t+4\sin 2t+2\sin 3t+\sin 4t}{3(1+\cos t)},$$

which, in particular, gives

t	$f(t)$	$\mathrm{Re}(4/5)_f(t)$
$\frac{\pi}{2}$	1.570963	1.3333333
$-\frac{\pi}{2}$	-1.570963	-1.3333333
$\frac{\pi}{3}$	1.570963	1.7320508
$-\frac{\pi}{3}$	-1.570963	-1.7320508
$\frac{\pi}{4}$	1.570963	1.8856181
$-\frac{\pi}{4}$	-1.570963	-1.8856181
$\frac{\pi}{5}$	1.570963	1.8096081
$-\frac{\pi}{5}$	-1.570963	-1.8096081
$\frac{\pi}{6}$	1.570963	1.6666666
$-\frac{\pi}{6}$	-1.570963	-1.6666666

(c). If $m=4$ and $\pi_{4,0}=\pi_{4,1}=\pi_{4,2}=\pi_{4,3}=\pi_{4,4}=0$, then

$$\mathrm{Re}(4/5)_f(t)=\sum_{\nu=1}^{4}\frac{1+(-1)^{\nu+1}}{\nu}\sin[\nu t]=2\left(\sin t+\frac{\sin 3t}{3}\right)$$

and

$$\mathrm{Re}(4/5)_f\left(\frac{\pi}{2}\right)=1.3333333,\quad \mathrm{Re}(4/5)_f\left(-\frac{\pi}{2}\right)=-1.3333333,$$

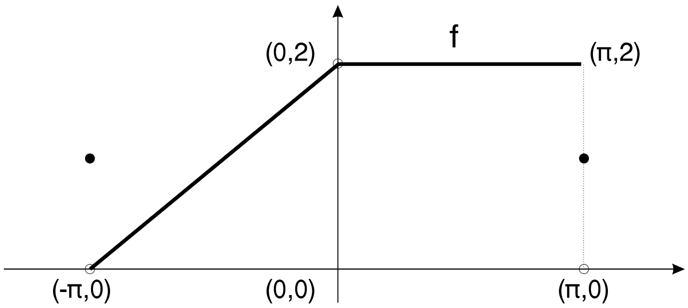
$$\mathrm{Re}(4/5)_f\left(\frac{\pi}{3}\right)=1.7320508,\quad \mathrm{Re}(4/5)_f\left(-\frac{\pi}{3}\right)=-1.7320508,$$

$$\mathrm{Re}(4/5)_f\left(\frac{\pi}{4}\right)=1.8856178,\quad \mathrm{Re}(4/5)_f\left(-\frac{\pi}{4}\right)=-1.8856178,$$

$$\mathrm{Re}(4/5)_f\left(\frac{\pi}{5}\right)=1.809608,\quad \mathrm{Re}(4/5)_f\left(-\frac{\pi}{5}\right)=-1.809608,$$

$$\mathrm{Re}(4/5)_f\left(\frac{\pi}{6}\right)=1.6666666,\quad \mathrm{Re}(4/5)_f\left(-\frac{\pi}{6}\right)=-1.6666666.$$

Example 1.4.7. Let f be the following function:



$$f(t)=\begin{cases}1 & ,if\ t=-\pi\\ 2\left(\dfrac{t}{\pi}+1\right) & ,if\ -\pi<t\le 0\\ 2 & ,if\ 0\le t<\pi\\ 1 & ,if\ t=\pi\end{cases}$$

As it is easily verified, the Fourier representation of f is given by

$$F(t)=\frac{3}{2}+2\sum_{\nu=1}^{\infty}\left[\frac{1-(-1)^{\nu}}{(\nu\pi)^2}\cos(\nu t)+\frac{(-1)^{\nu+1}}{(\nu\pi)}\sin(\nu t)\right]$$
$$=\frac{3}{2}+\sum_{\substack{\nu=-\infty\\(\nu\neq 0)}}^{\infty}\left[\frac{1-(-1)^{\nu}}{(\nu\pi)^2}+i\frac{(-1)^{\nu}}{\nu\pi}\right]e^{i\nu t}$$

$(-\pi\le t\le\pi)$. Let us define the $\mathbb{C}$-linear functional

$$T_f : \mathbf{P(C)} \to \mathbf{C} : x^\nu \mapsto T_f\left(x^\nu\right) = \begin{cases} \dfrac{3}{2} & , if\ \nu = 0 \\ \dfrac{1-(-1)^\nu}{(\nu\pi)^2} + i\dfrac{(-1)^\nu}{\nu\pi} & , if\ \nu = 1,2,3,... \end{cases}$$

For any complex triangular matrix

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m\geq 0, 0\leq k\leq m}$$

and any $m \geq 0$, a Padé-type approximant to f is a function

$$\mathrm{Re}(m/m+1)_f(t) = 2\,\mathrm{Re}\left[\frac{e^{-it}T_f\left(\dfrac{V_{m+1}\left(e^{-it}\right)-V_{m+1}(x)}{e^{-it}-x}\right)}{V_{m+1}\left(e^{-it}\right)}\right] - c_0 \quad (-\pi \leq t \leq \pi),$$

where

$$V_{m+1}(x) = \gamma\prod_{k=0}^{m}\left(x - \pi_{m,k}\right)$$

is the generating polynomial of this approximation ($\gamma \in \mathbf{C} - \{0\}$).

(a). If $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = 0,\ \pi_{3,3} = -\dfrac{1}{2}$, then

$$V_4(x) = x^4 + \frac{x^3}{2},$$

$$e^{-it}T_f\left(\frac{V_4\left(e^{-it}\right)-V_4(x)}{e^{-it}-x}\right)$$

$$= e^{-4it}\left(\frac{3}{2} + \left[\frac{3}{4} + \frac{2}{\pi^2} - \frac{i}{\pi}\right]e^{it} + \left[\frac{1}{\pi^2}\right]e^{2it} + \left[\frac{2}{9\pi^2} - \frac{i}{12\pi}\right]e^{3it}\right),$$

$$V_4\left(e^{-it}\right) = e^{-4it}\left(1 + \frac{e^{it}}{2}\right).$$

Thus,

$$\mathrm{Re}(3/4)_f(t) = \frac{4}{5+4\cos t}$$
$$\{3.9526423 + 0.6366197\sin t + 3.5066059\cos t + 0.251163\cos 2t$$
$$-0.1061032\sin 3t + 0.0450316\cos 3t - 0.0506655\sin 4t\} - 1.5\,.$$

In particular,

t	$f(t)$	$\mathrm{Re}(3/4)_f(t)$
0	2.0000000	1.9468554
π	1.0000000	1.1086712
$-\pi$	1.0000000	1.1086712
$\frac{\pi}{2}$	2.0000000	2.0553617
$-\frac{\pi}{2}$	1.0000000	0.9113552
$\frac{\pi}{3}$	2.0000000	2.0031648
$-\frac{\pi}{3}$	1.3333333	1.3730748
$\frac{\pi}{4}$	2.0000000	1.9619859
$-\frac{\pi}{4}$	1.5000000	1.5786325
$\frac{\pi}{5}$	2.0000000	1.9466674
$-\frac{\pi}{5}$	1.6000000	1.596714

(b). If $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = \pi_{3,3} = 0$, then

$V_4(x) = x^4$

and

$$\mathrm{Re}(3/4)_f(t) = \frac{3}{2} + 2\sum_{\nu=1}^{3}\left[\frac{1-(-1)^{\nu}}{(\nu\pi)^2}\cos(\nu t) + \frac{(-1)^{\nu+1}}{\nu\pi}\sin(\nu t)\right]$$

$$= 1.5 + 0.6366197\sin t + 0.4052847\cos t - 0.3183098\sin 2t + 0.2122065\sin 3t + 0.0450316\cos 3t\ .$$

Indicatively, we have

t	$f(t)$	$\mathrm{Re}(3/4)_f(t)$
0	2.0000000	1.9503163
π	1.0000000	1.0496837
$-\pi$	1.0000000	1.0496837
$\frac{\pi}{2}$	2.0000000	1.9244132
$-\frac{\pi}{2}$	1.0000000	1.0755868
$\frac{\pi}{3}$	2.0000000	1.9332752
$-\frac{\pi}{3}$	1.3333333	1.3819462
$\frac{\pi}{4}$	2.0000000	2.0366381
$-\frac{\pi}{4}$	1.5000000	1.4728366
$\frac{\pi}{5}$	2.0000000	2.2890725
$-\frac{\pi}{5}$	1.6000000	1.5406813

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Example 1.4.8. For $a \in \mathbb{R}$, let f_a be the real-valued function

$$f_a(t) = \sinh(a t) \qquad (t \in \mathbb{R}).$$

The Fourier series $F_a(t)$ of f_a into $(-\pi, \pi)$ is given by

$$F_a(t) = \frac{2\sinh(a\pi)}{\pi} \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}\nu}{\nu^2 + a^2} \sin(\nu t) = \sum_{\substack{\nu=-\infty \\ (\nu \neq 0)}}^{\infty} \frac{(-1)^{\nu} i\, \nu \sinh(a\pi)}{\pi(\nu^2 + a^2)} e^{i\nu t}.$$

Consider the $\mathbb{C}$-linear functional T_{f_a} associated with f_a:

$$T_{f_a} : \mathbb{P}(\mathbb{C}) \to \mathbb{C} : x^{\nu} \mapsto T_{f_a}(x^{\nu}) = c_{\nu}^{(a)} := \begin{cases} 0, if\ \nu = 0 \\ \dfrac{(-1)^{\nu} i\ \nu \sinh(a\pi)}{\pi(\nu^2 + a^2)}, if\ \nu = 1,2,3,... \end{cases}$$

If

$$\mathrm{M} = (\pi_{m,k})_{m \geq 0, 0 \leq k \leq m}$$

is a complex infinite triangular matrix, then, for any $m \geq 0$, a Padé-type approximant to $f_a(t)$ is a function

$$\mathrm{Re}(m/m+1)_{f_a}(t) = 2\,\mathrm{Re}\left[\frac{e^{-it} T_{f_a}\left(\dfrac{V_{m+1}(e^{-it}) - V_{m+1}(x)}{e^{-it} - x} \right)}{V_{m+1}(e^{-it})} \right] \quad (-\pi \leq t \leq \pi).$$

As usually,

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} (x - \pi_{m,k})$$

is the generating polynomial of this approximation ($\gamma \in \mathbb{C} - \{0\}$).

(a). If $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = \pi_{3,3} = 0$, then

$V_4(x) = x^4$

and

$$\mathrm{Re}(3/4)_{f_a}(t) = \frac{2\sinh(a\pi)}{\pi}\left(\frac{1}{1+a^2}\sin t - \frac{2}{4+a^2}\sin 2t + \frac{3}{9+a^2}\sin 3t\right).$$

Hence,

a	1/2		2	
t	$f_{1/2}(t)$	$\mathrm{Re}(3/4)_{f_{1/2}}(t)$	$f_2(t)$	$\mathrm{Re}(3/4)_{f_2}(t)$
0	0.0000000	0.0000000	0.000000	0.000000
$-\frac{\pi}{6}$	-0.2648002	-0.2954574	-1.249367	-19.476309
$\frac{\pi}{5}$	0.3193525	0.308832	1.614488	16.920309
$\frac{\pi}{4}$	0.4028703	0.3025888	2.3012989	9.3066229
$\frac{\pi}{3}$	0.5478534	0.2660742	3.9986913	-7.3807882
$\frac{\pi}{2}$	0.8686709	0.4436537	11.548739	-5.2446673

At first glance, the numerical results in the above table show an insufficiency status for our choice (at least for the case $a = 2$). However, for $\pi_{m,0} = \ldots = \pi_{m,m} = 0$ and m enough

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large, the corresponding Padé-type approximation results reveal sufficiently efficient, because of the coincidence of our approximation with the finite partial sum consisting in the first m terms of the Fourier series $F_a(t)$.

(b). If $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = 0, \pi_{3,3} = -\frac{2}{3}$, then

$$V_4(x) = x^4 + \frac{2}{3}x^3,$$

$$e^{-it}T_{f_a}\left(\frac{V_4(e^{-it}) - V_4(x)}{e^{-it} - x}\right) = e^{-4it}\left(\left[c_1^{(a)}\right]e^{it} + \left[c_2^{(a)} + \frac{2}{3}c_1^{(a)}\right]e^{2it} + \left[c_3^{(a)} + \frac{2}{3}c_2^{(a)}\right]e^{3it}\right),$$

$$V_4(e^{-it}) = e^{-4it}\left(1 + \frac{2}{3}e^{it}\right).$$

Thus,

$$\begin{aligned}\operatorname{Re} r(3/4)_{f_a}(t) = \frac{2\sinh(a\pi)}{\pi(13 + 12\cos t)}\Bigg(&\left[\frac{13}{1+a^2} - \frac{12}{4+a^2}\right]\sin t \\ &+ \left[\frac{6}{1+a^2} - \frac{26}{4+a^2} + \frac{18}{9+a^2}\right]\sin 2t \\ &+ \left[\frac{27}{9+a^2} - \frac{12}{4+a^2}\right]\sin 3t\Bigg).\end{aligned}$$

It follows that

a	$1/2$		2	
t	$f_{1/2}(t)$	$\mathrm{Re}(3/4)_{f_{1/2}}(t)$	$f_2(t)$	$\mathrm{Re}(3/4)_{f_2}(t)$
0	0.0000000	0.0000000	0.0000000	0.0000000
$-\frac{\pi}{6}$	-0.22648002	-0.2773082	-1.249367	-2.0063189
$\frac{\pi}{5}$	0.3193525	0.3317165	1.614488	2.1108546
$\frac{\pi}{4}$	0.4028703	0.4127542	2.3012989	2.0641907
$\frac{\pi}{3}$	0.5478534	0.7364422	3.9986913	1.6883143
$\frac{\pi}{2}$	0.8686709	0.8430904	11.548739	3.4292061

All these results seem to be enough successful (at least for the case $a=\frac{1}{2}$). But, on the other hand, some unexpected difficulties appear: if, for example, $a=-3$ then

$$\mathrm{Re}(3/4)_{f_{-3}}\left(\frac{\pi}{4}\right)=25.194418 \quad \text{while} \quad f_{-3}\left(\frac{\pi}{4}\right)=-5.2279719\,,$$

$$\mathrm{Re}(3/4)_{f_{-3}}\left(\frac{\pi}{3}\right)=2.0744719 \quad \text{while} \quad f_{-3}\left[\frac{\pi}{3}\right]=-11.548739\,.$$

Obviously, the variation of the real parameter a may cause spectacular perturbations in the behavior accuracy of our approximation results and, therefore, we must seek for constructing more satisfactory approximants in the generalized Padé-type sense ([48]).

1.4.2. Accelerating the Convergence of Functional Sequences

We shall now see how Padé-type approximants to continuous $2\pi-$periodic real-valued functions may accelerate the convergence of functional sequences. More precisely, we shall study the assumptions under which, for every sequence of functions converging to a real-valued continuous $2\pi-$periodic function on $[-\pi,\pi]$, there is always a Padé-type approximation sequence converging point-wise to that function faster than the first sequence. This property, due to the free choice of the interpolation points $\pi_{m,k}$, permits us to construct better approximations to continuous functions.

Before entering into more details, let me mention some well known results. Denoting by $(C[-\pi,\pi],\langle\cdot,\cdot\rangle)$ the space $C[-\pi,\pi]$ of all continuous real-valued functions defined on the interval $[-\pi,\pi]$, endowed with the usual scalar product

$$\langle g,h\rangle := \int_{-\pi}^{\pi} g(s)h(s)ds$$

and norm

$$\|g\|_2 := |\langle g,g\rangle|^{\frac{1}{2}} \quad (g,h\in C[-\pi,\pi]),$$

it is easily verified that $(C[-\pi,\pi],\langle\cdot,\cdot\rangle)$ is a not complete prehilbertian space. If X_{2n+1} is the $(2n+1)-$dimensional subspace of $(C[-\pi,\pi],\langle\cdot,\cdot\rangle)$ consisting of all trigonometric real-valued polynomials $p_n(t)$ with degree n, then the family

$$\{\varphi_0(t)=1,\varphi_1(t)=\cos t,\psi_1(t)=\sin t,\varphi_2(t)=\cos(2t),\psi_2(t)=\sin(2t),\ldots \\ \varphi_n(t)=\cos(nt),\psi_n(t)=\sin(nt)\}$$

is an orthonormal system in X_{2n+1}, and there exists always an element $\tilde{p}_n(t)$ of minimum distance from a fixed point $u(t)\in C[-\pi,\pi]$:

$$\tilde{p}_n(t) = \sum_{\nu=0}^{n} \alpha_\nu \varphi_\nu(t) + \sum_{\nu=1}^{n} \beta_\nu \psi_\nu(t)$$

with

$$\alpha_\nu = \frac{\langle \varphi_\nu, g \rangle}{\|\varphi_\nu\|_2^2} \quad \text{and} \quad \beta_\nu = \frac{\langle \psi_\nu, g \rangle}{\|\psi_\nu\|_2^2}.$$

The polynomial $\tilde{p}_n(t)$ is the orthogonal projection of $g(t)$ into X_{2n+1} and is called *the best approximation of* $g(t)$ *into* X_{2n+1}. Observe that, since the system

$$\{\varphi_0, \varphi_1, \psi_1, ,..., \varphi_n, \psi_n, ...\}$$

is fundamental in $(C[-\pi, \pi], \langle \cdot, \cdot \rangle)$, the orthogonal projection operator F_n of $(C[-\pi, \pi], \langle \cdot, \cdot \rangle)$ onto X_{2n+1} satisfies

$$\lim_{n\to\infty} \|F_n(g) - g\|_2 = \lim_{n\to\infty} \| \tilde{p}_n - g\|_2 = 0.$$

However, there exists a function $g(t) \in C[-\pi, \pi]$ with unbounded orthogonal projection sequence

$$\{F_n(g) : n \geq 0\}.$$

Similarly, if we consider the space $C[-\pi, \pi]$ endowed with the uniform norm

$$\|g\|_\infty = \sup_{t\in[-\pi,\pi]} |g(t)|$$

and the class $\mathfrak{R}_{n,m}[-\pi, \pi]$ of all rational functions

$$p_n(t)/q_m(t)$$

(n is the degree of $p_n(t)$ and m is the degree of $q_m(t) > 0$), then for any function $g(t) \in C[-\pi, \pi]$ one can associate at least one best rational approximation $\tilde{r}_{n,m}(t)$ from the class $\mathfrak{R}_{n,m}[-\pi, \pi]$. In both cases, the approximations are global and with respect to some norm.

One might hope that if we seek to approximate point-wise a real-valued continuous $2\pi-$periodic function, then the expected results will be presented earlier. In other words, one simply looks for other ways to recapture quickly a continuous $2\pi-$periodic real-valued function from a few known Fourier coefficients of this function. One such method is explored by means of Padé-type approximation and requires some preparatory material.

At first, by using techniques similar to those proposed by Bromwich and Clark in [27] and [33] (see also [18]), we prove the following

Proposition 1.4.9. *Let* Δ *be the operator of differences. Let also*

$$\{x_m : m = 0,1,2,...\} \quad \textit{and} \quad \{y_m : m = 0,1,2,...\}$$

be two sequences of real numbers satisfying

$$x_m \neq y_m \ \textit{ and } \ \lim_{m\to\infty} y_m = 0 .$$

Suppose $\{y_m : m = 0,1,2,...\}$ *is strictly monotone. If*

$$\lim_{m\to\infty}[\Delta x_m / \Delta y_m] = 0 ,$$

then

$$\lim_{m\to\infty} x_m = x \in \mathbb{R} \quad \textit{and} \quad \lim_{m\to\infty}[(x_m - x)/y_m] = 0 .$$

Proof. Let $\varepsilon > 0$. Without loss of generality, we can assume that the sequence $\{y_m : m = 0,1,2,...\}$ is strictly decreasing. (The case $\{y_m : m = 0,1,2,...\}$: strictly increasing is similar.) Since

$$\lim_{m\to\infty}[\Delta x_m / \Delta y_m] = 0 ,$$

there is a $M_0 > 0$ such that

$$-\varepsilon < (\Delta x_m / \Delta y_m) < \varepsilon \text{ for any } m \geq M_0 .$$

Since $\Delta y_m < 0$, these inequalities can be rewritten as

$$-\varepsilon(y_m - y_{m+1}) < x_m - x_{m+1} < \varepsilon(y_m - y_{m+1}) \quad \text{for any } m \geq M_0 .$$

Let us replace the index m by $m+1, m+2,..., m+p-1$. We have

$$-\varepsilon(y_m - y_{m+1}) < x_m - x_{m+1} < \varepsilon(y_m - y_{m+1})$$
$$-\varepsilon(y_{m+1} - y_{m+2}) < x_{m+1} - x_{m+2} < \varepsilon(y_{m+1} - y_{m+2})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$-\varepsilon(y_{m+p-1} - y_{m+p}) < x_{m+p-1} - x_{m+p} < \varepsilon(y_{m+p-1} - y_{m+p})$$

for any $m \geq M_0$. Adding these inequalities, we get

$$-\varepsilon(y_m - y_{m+p}) < x_m - x_{m+p} < \varepsilon(y_m - y_{m+p}) \text{ for any } m \geq M_0 .$$

Of course, we can suppose $y_m - y_{m+p} < 1$ and therefore obtain

$$-\varepsilon < x_m - x_{m+p} < \varepsilon \text{ for any } m \geq M_0 .$$

It follows that $\{x_m : m = 0,1,2,...\}$ is a Cauchy sequence. As the real field $\mathbb{R}$ is complete, this sequence converges to a limit in $\mathbb{R}$, say x. Letting now $p \to \infty$ in the inequalities

$$-\varepsilon(y_m - y_{m+p}) < x_m - x_{m+p} < \varepsilon(y_m - y_{m+p}) \quad (m \geq M_0),$$

we get

$$-\varepsilon\, y_m < x_m - x < \varepsilon\, y_m , \quad \text{for any } m \geq M_0 ,$$

which implies that

$$\lim_{m\to\infty}[(x_m - x)/y_m].$$

The *Proof* is now complete.

Recall also that for two numerical real sequences $\{x_m : m = 0,1,2,...\}$ and $\{y_m : m = 0,1,2,...\}$ converging to x and y respectively, we say that

$$\{x_m : m = 0,1,2,...\} \textit{ converges faster than } \{y_m : m = 0,1,2,...\},$$

if

$$\lim_{m\to\infty}[(x_m - x)/(y_m - y)] = 0.$$

With this terminology, we can now quote a classical result which is a direct consequence of *Proposition 1.4.9*.

Corollary 1.4.10. *Let* Δ *be the operator of differences. Let also*

$$\{x_m : m = 0,1,2,...\} \qquad \textit{and} \qquad \{y_m : m = 0,1,2,...\}$$

be two sequences of real numbers converging respectively to $x \in \mathbb{R}$ *and* $y \in \mathbb{R}$. *Suppose* $\{y_m : m = 0,1,2,...\}$ *is strictly monotone. If*

$$\lim_{m\to\infty}[\Delta x_m / \Delta y_m] = 0,$$

then the sequence $\{x_m : m = 0,1,2,...\}$ *converges faster than the sequence* $\{y_m : m = 0,1,2,...\}$.

In [25], C. Brezinski showed that *if*

$$\overline{\lim}_{m\to\infty}|\Delta x_m|^{\frac{1}{m}} = R < r = \lim_{m\to\infty}|\Delta y_m|^{\frac{1}{m}},$$

then the sequence $\{\Delta x_m : m = 0,1,2,...\}$ *converges faster than the sequence* $\{\Delta y_m : m = 0,1,2,...\}$:

$$\lim_{m\to\infty}[\Delta x_m / \Delta y_m] = 0.$$

The main criterion of *Corollary 1.4.10* can thus be rephrased as follows.

Corollary 1.4.11. *Let* Δ *be the operator of differences. Let also*

$$\{x_m : m = 0,1,2,...\} \qquad \textit{and} \qquad \{y_m : m = 0,1,2,...\}$$

be two sequences of real numbers converging respectively to $x \in \mathbb{R}$ *and* $y \in \mathbb{R}$. *If* $\{y_m : m = 0,1,2,...\}$ *is strictly monotone and*

$$\overline{\lim}_{m\to\infty} |\Delta x_m|^{\frac{1}{m}} = R < \lim_{m\to\infty} |\Delta y_m|^{\frac{1}{m}} = r,$$

then the sequence $\{x_m : m = 0,1,2,...\}$ *converges faster than the sequence* $\{y_m : m = 0,1,2,...\}$.

The following three results give us now some theoretical answers to the convergence acceleration problem by means of Padé-type approximants.

Theorem 1.4.12. *Suppose there is a constant* $K > 0$ *and an open neighborhood* U *of the unit circle into which the generating polynomials* $V_{m+1}(x)$ *of a Padé-type approximation satisfy*

$$K \leq |V_{m+1}(z)|, \textit{for any } z \in U \textit{ and any } m \textit{ sufficiently large}.$$

Further, assume that the family $\{V_{m+1}(e^{is}) : m = 0,1,2,...\}$ *is orthonormal in* $L^2[-\pi,\pi]$. *If*

$$\overline{\lim}_{m\to\infty} \left\{ \sup_{|x|\leq 1} \left| \frac{V_{m+1}(x)V_{m+2}(e^{-it}) - V_{m+1}(e^{-it})V_{m+2}(x)}{1 - xe^{it}} \right| \right\}^{1/m} = R(t) \quad (t \in [-\pi,\pi]),$$

then, for any real-valued continuous $2\pi -$*periodic function* f *on* $[-\pi,\pi]$, *the corresponding sequence*

$$\{\mathrm{Re}(m/m+1)_f(t) : m = 0,1,2,...\}$$

of Padé-type approximants to f converges to $f(t)$ *faster than any strictly monotone converging sequence* $\{y_m : m = 0,1,2,...\}$ *satisfying*

$$\lim_{m\to\infty} |\Delta y_m|^{\frac{1}{m}} > R(t).$$

Proof. By *Theorem 1.3.21.(b)*, the sequence $\{\mathrm{Re}(m/m+1)_f(t) : m = 0,1,2,...\}$ converges to $f(t)$ everywhere in $[-\pi,\pi]$. Letting $t \in [-\pi,\pi]$ be fixed, we have

$$\left|\Delta\left(\mathrm{Re}(m/m+1)_f(t)\right)\right|^{\frac{1}{m}} \leq 2\left|T_f\left(G_{m+1}\left(x,e^{it}\right) - G_m\left(x,e^{it}\right)\right)\right|^{\frac{1}{m}}.$$

The continuity of the linear functional T_f implies now that

$$\left|\Delta\left(\mathrm{Re}(m/m+1)_f(t)\right)\right|^{\frac{1}{m}} \leq \left(\frac{\Im_f}{K}\right)^{\frac{1}{m}} \left[\sup_{|x|\leq 1}\left|\frac{V_{m+1}(x)V_{m+2}\left(e^{-it}\right) - V_{m+2}\left(e^{-it}\right)V_{m+1}(x)}{1-xe^{it}}\right|\right]^{\frac{1}{m}},$$

where the constant $\Im_f$ depends only on f. By passing in the upper limit, we obtain

$$\overline{\lim}_{m\to\infty}\left|\Delta\left(\mathrm{Re}(m/m+1)_f(t)\right)\right|^{\frac{1}{m}} \leq R(t).$$

Application of *Corollary 1.4.11* for the sequences $\{x_m = \mathrm{Re}(m/m+1)_f(t) : m = 0,1,2,...\}$ and $\{y_m : m = 0,1,2,...\}$ proves the *Theorem*.

Similarly, using *Corollary 1.3.22* instead of *Theorem 1.3.21*, we are leaded to

Corollary 1.4.13. *Let*

$$V_{m+1}(x)=\gamma\prod_{k=0}^{m}\left(x-\pi_{m,k}\right)=\sum_{k=0}^{m+1}b_k^{(m)}x^k \quad (m=0,1,2,\ldots)$$

be the generating polynomials of a Padé-type approximation such that

$$\sum_{k=0}^{m+1}\left|b_k^{(m)}\right|^2=\frac{1}{2\pi} \quad (m\geq 0)\,, \qquad \sum_{k=0}^{m+1}b_k^{(m)}\overline{b_k^{(n)}}=0 \quad (m<n)$$

and

$$\overline{\lim}_{m\to\infty}\left[\sum_{k=0}^{m+2}\sum_{\substack{\nu=0\\(\nu\neq k)}}^{m+1}\left|b_k^{(m+1)}b_\nu^{(m)}\right|\right]^{\frac{1}{m}}=R\,.$$

If there are two constants $\sigma<\infty$ *and* $c<1$ *fulfilling*

$$\sum_{k=0}^{m+1}\left|b_k^{(m)}\right|<\sigma \quad (m\geq 0) \quad \textit{and} \quad \left|\pi_{m,k}\right|<c \quad (m\geq 0, 0\leq k\leq m)\,,$$

then, for any real-valued continuous $2\pi-$*periodic function* f *on* $[-\pi,\pi]$, *the corresponding sequence*

$$\left\{\mathrm{Re}(m/m+1)_f(t):m=0,1,2,\ldots\right\}$$

of Padé-type approximants to f *converges to* $f(t)$ *everywhere in* $[-\pi,\pi]$, *faster than any strictly monotone converging sequence* $\{y_m:m=0,1,2,\ldots\}$ *satisfying*

$$\lim_{m\to\infty}\left|\Delta y_m\right|^{1/m}>R.$$

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For a *Proof* of this *Corollary*, we only note that

$$\left|V_{m+1}(x)V_{m+2}\left(e^{-it}\right)-V_{m+1}\left(e^{-it}\right)V_{m+2}(x)\right| \le \sum_{k=0}^{m+2}\sum_{\substack{\nu=0\\(\nu\neq k)}}^{m+1}\left|b_k^{(m+1)}b_\nu^{(m)}\right|\left|e^{itk}x^k-e^{it\nu}x^\nu\right|.$$

Hence

$$\overline{\lim}_{m\to\infty}\left\{\sup_{|x|\le 1}\left|\frac{V_{m+1}(x)V_{m+2}\left(e^{-it}\right)-V_{m+2}(x)V_{m+1}\left(e^{-it}\right)}{1-xe^{it}}\right|\right\}^{\frac{1}{m}}$$

$$\le \overline{\lim}_{m\to\infty}\left[\sum_{k=0}^{m+1}\sum_{\substack{\nu=0\\(\nu\neq k)}}^{m}\left|b_k^{(m+1)}b_\nu^{(m)}\right|\right]^{\frac{1}{m}} =: R\,,$$

and we may apply *Theorem 1.4.12*.

Theorem 1.4.14. *Assume that the generating polynomials* $V_{m+1}(x)$ *of a Padé-type approximation satisfy*

$$\lim_{m\to\infty}\frac{V_{m+1}(x)}{V_{m+1}\left(z^{-1}\right)}=0$$

compactly in an open set $\omega\subset\mathbb{C}^2$ *containing* $\left(\overline{D}\times D\right)\cup(\mathbb{C}\times\{0\})$. *If*

$$\overline{\lim}_{m\to\infty}\left\{\sup_{|x|\le 1}\left|\frac{V_{m+1}(x)V_{m+2}\left(e^{-it}\right)-V_{m+2}(x)V_{m+1}\left(e^{-it}\right)}{1-xe^{it}}\right|\right\}^{\frac{1}{m}} =: R(t)\quad (t\in[-\pi,\pi]),$$

then for any real-valued function f, *continuous and* $2\pi-$*periodic on* $[-\pi,\pi]$, *the corresponding sequence*

$$\left\{\mathrm{Re}(m/m+1)_f(t): m=0,1,2,\ldots\right\}$$

of Padé-type approximants to f *converges to* $f(t)$ *faster than any strictly monotone converging sequence* $\{y_m : m = 0,1,2,...\}$ *such that*

$$\lim_{m\to\infty} |\Delta y_m|^{\frac{1}{m}} > R(t).$$

The *Proof* of *Theorem 1.4.14* is exactly similar to that of *Theorem 1.4.12* except for the fact that here we must apply *Theorem 1.3.23.(b)* instead of *Theorem 1.3.21.(b)*.

1.4.3. Approximate Computation of Derivatives and Integrals

Suppose f is real continuous in the interval $-\pi \le t \le \pi$, where $f(-\pi) = f(\pi)$, and f' is piecewise continuous on the interval $-\pi \le t \le \pi$.

Then the Fourier series in the representation

$$F(t) = \sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu t}$$

is differentiable at each point $t \in (-\pi, \pi)$ at which $f''(t)$ exists:

$$F'(t) = \sum_{\nu=-\infty}^{\infty} c_\nu i\nu e^{i\nu t} \quad ([27]).$$

Defining the $\mathbb{C}$-linear functionals

$$T_f : \mathbb{P}(\mathbb{C}) \to \mathbb{C} \text{ and } T_{f'} : \mathbb{P}(\mathbb{C}) \to \mathbb{C}$$

by

$$T_f(x^\nu) = c_\nu \quad \text{and} \quad T_{f'}(x^\nu) = i\nu c_\nu \quad (\nu \ge 0),$$

respectively, it is easily seen that for any $p(x) \in \mathbb{P}(\mathbb{C})$ there holds

$$T_{f'}(p(x)) = T_f(ixp'(x)).$$

Thus, if the complex infinite triangular matrix

$$\mathrm{M} = (\pi_{m,k})_{m\geq 0, 0\leq k\leq m}$$

is given, then for any $m \geq 0$, a Padé-type approximant to $f'(t)$ is the function

$$\mathrm{Re}(m/m+1)_{f'}(t) = 2\,\mathrm{Re}\left[\frac{ie^{it}}{V_{m+1}(e^{-it})}T_f\left(x\left\{\frac{V_{m+1}(e^{-it}) - V_{m+1}(x)}{e^{-it} - x}\right\}'\right)\right],$$

or

$$\mathrm{Re}(m/m+1)_{f'}(t) = -2\,\mathrm{Re}\left[\frac{ie^{it}}{V_{m+1}(e^{-it})}T_f\left(\frac{xV'_{m+1}(x)}{e^{-it} - x}\right)\right.$$

$$\left.+\frac{ie^{it}}{V_{m+1}(e^{-it})}T_f\left(xe^{-it}\frac{V_{m+1}(e^{-it}) - V_{m+1}(x)}{(e^{-it} - x)^2}\right)\right].$$

As in the *Proofs* of *Theorem 1.3.9.(b)* and *Proposition 1.3.10.(b)*, one can show that the error of this approximation is given by the following formula:

$$\mathrm{Re}(m/m+1)_{f'}(t) - f'(t) = \frac{1}{\pi}\lim_{r\to 1}\mathrm{Re}\left[i\int_{-\pi}^{\pi}\frac{f(s)}{re^{it} - e^{is}}\frac{V'_{m+1}(e^{-is})}{V_{m+1}(r^{-1}e^{-it})}ds\right],$$

where the limit is uniform on $[-\pi,\pi]$. From *Cauchy's Integral Formula*, it follows that

$$\mathrm{Re}(m/m+1)_{f'}(t) - f'(t) = 0 \ ,$$

and therefore

$$\mathrm{Re}(m/m+1)_{f'}(t) = 0 \ ,$$

whenever $f(t)$ is a constant function in $[-\pi,\pi]$.

Integration of Fourier series is also possible under much more general conditions than those of differentiation: Let f be a real function that is piecewise continuous on the interval

$-\pi < t < \pi$; regardless of whether series $F(t)$ converges, the following equation is valid whenever $-\pi \leq t_0 \leq t \leq \pi$:

$$\int_{t_0}^{t} f(s)ds = c_0(t-t_0) + \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} \frac{c_\nu}{i\nu} e^{i\nu t} - \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} \frac{c_\nu}{i\nu} e^{i\nu t_0} \qquad ([27]).$$

Observe that

$$\int_{-\pi}^{\pi} f(s)ds = 2\pi c_0, \qquad \int_{0}^{\pi} f(s)ds = \pi c_0 \quad \text{and} \quad \int_{-\pi}^{0} f(s)ds = \pi c_0.$$

Define the $\mathbf{C}$-linear functional $T_{\int f} : \mathbf{P}(\mathbf{C}) \to \mathbf{C}$ by

$$T_{\int f}(x^\nu) = \begin{cases} \dfrac{c_\nu}{i\nu}, if\ \nu \geq 1 \\ 0, if\ \nu = 0 \end{cases}.$$

Then, for any $m \geq 0$, the definite integral $\int_{t_0}^{t} f(s)ds$ can be approximated in the Padé-type sense by the number

$$\mathrm{Re}(m/m+1)\left(\int_{t_0}^{t} f\right) := c_0(t-t_0) + 2\,\mathrm{Re}\left[\frac{e^{it}}{V_{m+1}(e^{-it})} T_{\int f}\left(\frac{V_{m+1}(e^{-it}) - V_{m+1}(x)}{e^{-it} - x}\right)\right.$$
$$\left. - \frac{e^{it_0}}{V_{m+1}(e^{-it_0})} T_{\int f}\left(\frac{V_{m+1}(e^{-it_0}) - V_{m+1}(x)}{e^{-it_0} - x}\right)\right].$$

Notice that

$$\mathrm{Re}(m/m+1)\left(\int_{-\pi}^{\pi} f\right) = \int_{-\pi}^{\pi} f(s)ds \quad \text{and} \quad \mathrm{Re}(m/m+1)\left(\int_{-\pi}^{0} f\right) = \int_{-\pi}^{0} f(s)ds.$$

The error is

$$\mathrm{Re}(m/m+1)\left(\int_{t_0}^{t} f\right) - \int_{t_0}^{t} f(s)ds = \lim_{r\to 1} 2\,\mathrm{Re}\left[\frac{1}{V_{m+1}(r^{-1}e^{-it})} T_{\int f}\left(\frac{V_{m+1}(x)}{xre^{it} - 1}\right)\right.$$
$$\left. - \frac{1}{V_{m+1}(r^{-1}e^{-it_0})} T_{\int f}\left(\frac{V_{m+1}(x)}{xre^{it_0} - 1}\right)\right].$$

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Chapter 2

Interpolation Methods for the Evaluation of a 2π-Periodic Finite Baire Measure and Integral Representations for Padé -Type Operators

Summary

In this *Chapter*, we will discuss the definition and effectiveness of Padé-type approximants to 2π-periodic finite Baire measures on [-π,π], as well as the convergence of a sequence of such approximants in the weak-star topology of measures. The next purpose of the *Chapter* is to look at an explicit form of Padé-type operators. To do so, we will consider representations of Padé-type approximants to harmonic, analytic, and L^p-functions by means of integral formulas, and then, we will define corresponding Padé-type operators. We will also study the basic properties of these integral operators and will prove convergence results.

Introduction

One of the most effective methods for the numerical solution of integral equations imposes replacement of the integral equation by an system of linear equations, using aquadrature formula.

Indeed, suppose we are given the equation

$$(E1) \qquad f(s)-\lambda\int_{-\pi}^{\pi}h(s,t)f(t)\phi(t)dt=g(s).$$

If we replace the integral, using a numerical interpolation formula of the form

$$\int_{-\pi}^{\pi} f(t)dt = \sum_{k=1}^{n} A_k f(t_k),$$

based on the points $t_1, t_2, \ldots, t_n$, and require $(E1)$ to be satisfied only at these points, then we obtain a system

$$f(t_j) - \lambda \sum_{k=1}^{n} A_k h(t_j, t_k) f(t_k) \phi(t_k) = g(t_j) \quad (j = 1, 2, \ldots, n).$$

Any solution of this system determines an approximate value for the required solution at the points $t_1, t_2, \ldots, t_n$.

It often happens that the integral in $(E1)$ is considered with respect to some finite measure μ on $[-\pi, \pi]$:

$$(E2) \qquad f(s) - \lambda \int_{-\pi}^{\pi} h(s,t) f(t) d\mu(t) = g(s).$$

Below, we shall be concerned with constructing a general approximation method for a large class of measures μ on $[-\pi, \pi]$, in such a way that one can approximate to an integral equation $(E2)$ by replacing it by an equation of type $(E1)$.

Let us begin with a finite real Baire measure μ on $[-\pi, \pi]$. For definiteness, we assume that μ is $2\pi-$periodic, i.e. if μ has a point mass at $-\pi$ or π, these masses must be the same: $\mu(\{-\pi\}) = \mu(\{\pi\})$. Then, μ can be regarded as a measure on the unit circle C, obtained by identifying $-\pi$ and π, and the Poisson integral $u(re^{it}) = u(z)$ of μ is a real-valued harmonic function in z. From the solution of Dirichlet's problem in the unit disk D, it follows that, when $r \to 1$, the measures

$$d\mu_r(t) = u_r(t)dt \qquad (u_r(t) = u(re^{it}))$$

converge to $d\mu(t)$ in the weak-star topology on measures.

By using interpolation methods, we shall seek for an effective approximation to $u_r(t)$. A natural approach to its solution is afforded by the ideas of Padé-type approximation. According to [42], the Padé-type approximants $\mathrm{Re}(m/m+1)_u(z)$ to the harmonic function $u(z)$ can be chosen in such a way to be harmonic real-valued function everywhere on D; their fundamental property is that the Fourier series expansion of the restriction $\mathrm{Re}(m/m+1)_{u_r}(t)$ of $\mathrm{Re}(m/m+1)_u(re^{it})$ to the circle C_r of radius $r<1$ matches the Fourier series expansion of the restriction $u_r(t)$ of $u(re^{it})$ to C_r up to the $\pm m^{th}$ Fourier term. We can therefore approximate $d\mu_r(t)$ by $\mathrm{Re}(m/m+1)_u(re^{it})dt$. When $r \to 1$, the measures $\mathrm{Re}(m/m+1)_u(re^{it})dt$ converge to the finite real measure

$$\mathrm{Re}(m/m+1)_\mu(t)dt := \mathrm{Re}(m/m+1)_u(e^{it})dt$$

in the weak-star topology on measures. The boundedness in L^1-norm of the family $\{\mathrm{Re}(m/m+1)_u(re^{it}) : 0 \le r < 1\}$ guarantees that the Fourier series expansion of the limit measure $\mathrm{Re}(m/m+1)_\mu$ matches the Fourier series expansion of μ up to the $\pm m^{th}-$order's Fourier-Stieltjes term. The measure $\mathrm{Re}(m/m+1)_\mu(t)dt$ will be called a *Padé-type approximant to* $d\mu(t)$. The integral equation

$$f(s) - \lambda \int_{-\pi}^{\pi} h(s,t) f(t)\, \mathrm{Re}(m/m+1)_\mu(t)\, dt = g(s)$$

is a *Padé-type approximate equation to* $(E2)$.

To judge the effectiveness of this method, and the extent to which it can be justified, the method need to be investigated theoretically. So, in the first *Section* of this *Chapter*, we discuss the definition and effectiveness of a Padé-type approximation to a finite Baire measure.

The second *Section* deals with the representation form of Padé-type approximants by means of integral formulas and the consideration of corresponding Padé-type operators.

Let f be a function analytic in the open unit disk D, with Taylor power series expansion

$$\sum_{\nu=0}^{\infty} a_\nu z^\nu$$

Let also T_f be the linear functional defined on the space of complex polynomials by $T_f\left(x^\nu\right)=a_\nu$. By *Cauchy's Integral Formula* and *Hahn-Banach Theorem*, the functional T_f can be extended to the space $A(\overline{D})$ of all functions which are analytic in D and continuous in the open neighborhood of $\overline{D}$ ([40]). In particular, we have

$$f(z)=T_f\left((1-xz)^{-1}\right) \quad \text{for any } z\in D.$$

Now, let $V_{m+1}(x)$ be an arbitrary polynomial of degree $m+1$, with distinct zeros $\pi_1,\pi_2,\ldots,\pi_n$ of respective multiplicities $(m_1+1),(m_2+1),\ldots,(m_n+1)$. If $(m_1+1)+\cdots+(m_n+1)=m+1$, denote by $I(V_{m+1})$ the linear operator mapping each $h(x)\in A(\overline{D})$ to its Hermite interpolation polynomial G_{m+1} of degree at most m defined by

$$h^{(j)}(\pi_i)=G_{m+1}^{(j)}(\pi_i) \quad \text{for } i=1,2,\ldots,n \text{ and } j=0,1,\ldots,m.$$

If $h(x,z)\equiv(1-xz)^{-1}$, then $T_f\left(G_{m+1}(x,z)\right)$ is the so-called *Padé-type approximant to* $f(z)$ with generating polynomial $V_{m+1}(x)$. It is a rational function with numerator of degree m and denominator of degree $m+1$, denoted by $(m/m+1)_f(z)$ and such that

$$f(z)-(m/m+1)_f(z)=O\left(z^{m+1}\right), \text{ if } |z|<\min\left\{\left(1/|\pi_1|\right),\ldots,\left(1/|\pi_n|\right)\right\} \quad ([20]).$$

Making use of the duality notation, we can also write

$$(m/m+1)_f(z)==T_f\left(G_{m+1}(x,z)\right)=\left\langle T_f,[I(v_{m+1})](1-xz)^{-1}\right\rangle=\left\langle\left[I^*(v_{m+1})\right](T_f)(1-xz)^{-1}\right\rangle.$$

Note that if $V_{m+1}(x)$ is identical to the orthogonal polynomial $q_{m+1}(x)$ with respect to T_f, that is $T_f\left(x^\nu q_{m+1}(x)\right)=0$ whenever $\nu=0,1,2,\ldots,m$, then the Padé-type approximant

$(m/m+1)_f(z)$ becomes the classical Padé approximant $[m/m+1]_f(z)$ to $f(z)$, in the sense that

$$f(z)-[m/m+1]_f(z)=\mathrm{O}\left(z^{2m+2}\right), \text{ if } |z|<\min\{(1/|\pi_1|),\dots,(1/|\pi_n|)\} \text{ ([21]).}$$

In [26], Brezinski showed that the operator which maps f to $(m/m+1)_f$ can be understood as the mapping of $A^*(\overline{D})$ into itself which maps T_f into $[I^*(V_{m+1})](T_f)$. This mapping, which depends on the generating polynomial $V_{m+1}(x)$, is called *the Padé-type operator for the space* $\mathcal{O}(D)$ *of all analytic functions on* D and it is exactly the operator $I^*(V_{m+1})$. If $V_{m+1}(x)$ does not depend on T_f, then $I^*(V_{m+1})$ is linear. But for Padé approximants, since $V_{m+1}(x)$ is the orthogonal polynomial $q_{m+1}(x)$ of degree $m+1$ with respect to the functional T_f, then $V_{m+1}(x)$ depends on T_f, and the linearity property only holds if the first $2m+2$ moments of both functionals are the same since, then, both orthogonal polynomials of degree $m+1$ will be the same. The aim of the second *Section* is to look at an explicit form of the Padé-type operator by means of integral representations. The first *Paragraph* of this *Section* deals with integral representations of Padé-type approximants to real-valued L^2 or harmonic functions and, thus, with expressions of Padé-type operators for the spaces $L^2_{\mathbb{R}}(C)$(of all real-valued L^2 functions on C), $L^2_{\mathbb{R},(2\pi-per.)}[-\pi,\pi]$ (of all real-valued 2π-periodic L^2 functions on $[-\pi,\pi]$), and $H_{\mathbb{R}}(D)$ (of all real-valued harmonic functions on D). In *Paragraph 2.2.2*, we define and give the explicit form of the composed Padé-type operators for the spaces $L^2_{\mathbb{C}}(C)$ of all complex-valued L^2 functions on C, $L^2_{\mathbb{C},(2n-per.)}[-\pi,\pi]$ of all complex-valued $2\pi-$periodic L^2 functions on $[-\pi,\pi]$, and $H_{\mathbb{C}}(D)$ of all complex-valued harmonic functions on D. Since $\mathcal{O}(D)\subset H_{\mathbb{C}}(D)$, we thus obtain the desired explicit form of $I^*(V_{m+1})$.

2.1. Interpolation Methods for the Evaluation of a Measure

2.1.1. Padé-type Approximation to Finite Baire Measures

In this *Paragraph*, we shall introduce Padé-type approximation to measures on $[-\pi,\pi]$.

Let μ be any finite real Baire measure on $[-\pi,\pi]$. Since $[-\pi,\pi]$ is a closed subset of Euclidean space, the Baire and Borel subsets of $[-\pi,\pi]$ coincide, so μ may also be identified with a real Baire measure.

We can define the Fourier-Stieltjes coefficients of μ by

$$\sigma_\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\nu\theta} d\mu(\theta) \quad (\nu = 0,\pm 1,\pm 2,...)$$

and the associated Fourier series. We might expect a sequence of measures to converge to μ in the weak-star topology on measures, but the measure μ must have period 2π. This means that if μ has a point mass at $-\pi$ and π, these masses must be the same, i.e.

$$\mu(\{-\pi\}) = \mu(\{\pi\}).$$

A better way to formulate this condition is that μ is really a measure on the circle obtained by identifying $-\pi$ and π. If

$$u\left(re^{it}\right) = u_r(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t-\theta)\,d\mu(\theta) = \sum_{\nu=-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\nu\theta} d\mu(\theta)\right) r^{|\nu|} e^{i\nu t}$$

$$= \sum_{\nu=-\infty}^{\infty} \sigma_\nu\, r^{|\nu|}\, e^{i\nu t}$$

$(0 \le r < 1, -\pi \le t \le \pi)$ is the Poisson integral of the measure μ, then the function $u(z)$ is harmonic real-valued at z in the disk and the measures $d\mu_r(t) = u_r(t)dt$ converge to $d\mu(t)$ in the weak-star topology on measures, i.e.

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$$\lim_{r\to 1}\int_{-\pi}^{\pi} f(t)\,d\mu_r(t) = \int_{-\pi}^{\pi} f(t)\,d\mu(t)$$

for any real-valued continuous function f on $[-\pi,\pi]$.

In order to define Padé-type approximation to μ, we consider again the infinite triangular interpolation matrix

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m\geq 0, 0\leq k\leq m}$$

with complex entries $\pi_{m,k}$ satisfying $\left|\pi_{m,k}\right| < 1$ ($m \geq 0, 0 \leq k \leq m$). For any m and any fixed $z \in \mathbb{C} - \left\{\pi_{m,k}^{-1} : k = 0,1,\ldots,m\right\}$, let us consider the unique polynomial $G_m(x,z)$ of degree at most m which interpolates the function $(1-xz)^{-1}$ at

$$x = \pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}\,.$$

Let us define the corresponding Padé-type approximant to $u(z) = u\left(re^{it}\right)$ with generating polynomial

$$V_{m+1}(x) = \gamma\prod_{k=0}^{m}\left(x - \pi_{m,k}\right)$$

by

$$\operatorname{Re}(m/m+1)_u\left(r\,e^{it}\right) = 2\operatorname{Re}\left[\frac{\tilde{W}_m\left(r\,e^{it}\right)}{\tilde{V}_{m+1}\left(r\,e^{it}\right)}\right] - \sigma_0 = 2\operatorname{Re}T_\mu\left(G_m\left(x, r\,e^{it}\right)\right) - \sigma_0\,.$$

As usually, we have used the notation

$$\tilde{W}_m\left(r.e^{it}\right) := r^m\, e^{imt}\, T_\mu\left(\frac{V_{m+1}\left(r^{-1}\,e^{-it}\right)}{r^{-1}\,e^{-it} - x}\right), \qquad \tilde{V}_{m+1}\left(r\,e^{it}\right) := r^{m+1}\, e^{i(m+1)t}\, V_{m+1}\left(r^{-1}\,e^{-it}\right).$$

T_μ is the linear functional

$$T_{\mu}:\mathrm{P}(\mathbb{C})\rightarrow\mathbb{C}: x^{\nu}\mapsto T_{\mu}\left(x^{\nu}\right):=\sigma_{\nu}\quad(\nu=0,1,2,\ldots).$$

Then

$$\operatorname{Re}(m/m+1)_{u}(z)=\operatorname{Re}(m/m+1)_{u}\left(r\,e^{it}\right)$$

is a harmonic real-valued function in the open unit disk. Further, if the Fourier series representation of $\operatorname{Re}(m/m+1)_{u}\left(r\,e^{it}\right)$ is

$$\operatorname{Re}(m/m+1)_{u}\left(re^{it}\right)=\sum_{\nu=-\infty}^{\infty}d_{\nu}^{(m)}r^{|\nu|}e^{i\nu t},$$

the choice $\left|\pi_{m,k}\right|<1$ for any $k\leq m$ implies that

$$d_{\nu}^{(m)}=\sigma_{\nu}\quad\text{ for any }\nu=0,\pm1,\pm2,\ldots,\pm m.$$

Clearly, when $r\rightarrow1$, the measures $\operatorname{Re}(m/m+1)_{u}\left(r\,e^{it}\right)d\,t$ converge to the finite real measure on $[-\pi,\pi]$:

$$\begin{aligned}\operatorname{Re}(m/m+1)_{\mu}(t)dt&:=\operatorname{Re}(m/m+1)_{u}\left(r\,e^{it}\right)dt\\&=\left\{2\operatorname{Re}\left(\tilde{W}_{m}(e^{it})/\tilde{\mathrm{V}}_{m+1}(e^{it})\right)-\sigma_{0}\right\}dt=\left\{2\operatorname{Re}T_{\mu}\left(G_{m}\left(x,e^{it}\right)\right)-\sigma_{0}\right\}dt,\end{aligned}$$

in the weak-star topology on measures, and, moreover, the boundedness in $L^{1}-$norm of the family $\left\{\operatorname{Re}(m/m+1)_{u}\left(r\,e^{it}\right):\ 0\leq r<1\right\}$ guarantees that the Fourier series expansion of the limit measure $\operatorname{Re}(m/m+1)_{\mu}$ is given by

$$\sum_{\nu=-\infty}^{\infty}d_{\nu}^{(m)}e^{i\nu t}.$$

Definition 2.1.1. *The finite real Baire measure on* $[-\pi,\pi]$

$$\operatorname{Re}(m/m+1)_{\mu}dt$$

is called Padé-type approximant to μ*, with generating polynomial* $V_{m+1}(x)$.

According to our discussion above, if only a few Fourier-Stieltjes coefficients of a finite real Baire measure μ are known, one can approximate μ by its Padé-type approximants, in the sense that if $|\pi_{m,k}| < 1$ for any $k \leq m$, then

$$d_{\nu}^{(m)} = \sigma_{\nu}$$

whenever $-m \leq \nu \leq m$.

Let us study the error formula.

Theorem 2.1.2. *For any* $m \geq 0$, *it holds*

$$\operatorname{Re}(m/m+1)_{\mu}(t)dt - d\mu(t) = \lim_{r\to 1}\left\{2\operatorname{Re}\left(\frac{1}{\mathrm{V}_{m+1}(r^{-1}e^{-it})} - T_{\mu}\left(\frac{\mathrm{V}_{m+1}(x)}{x\,r\,e^{it}-1}\right)\right)d\,t\right\}$$

$$= \lim_{r\to 1}\left\{2\operatorname{Re}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{re^{i(t-s)}-1}\frac{V_{m+1}(e^{-is})}{V_{m+1}(r^{-1}e^{-it})}d\,\mu(s)\right)d\,t\right\}$$

in the weak-star topology on measures.

Proof. Let $\{0 \leq r_n < 1 : n = 0,1,2,\ldots\}$ be such that

$$\lim_{n\to\infty} r_n = 1.$$

As it is mentioned above

$$\lim_{n\to\infty} d\mu_{r_n} = d\mu$$

and

$$\lim_{n\to\infty} \operatorname{Re}(m/m+1)_u\left(r_n e^{it}\right)dt = \operatorname{Re}(m/m+1)_\mu(t)dt\,,$$

in the weak-star topology on measures. It follows that

$$\operatorname{Re}(m/m+1)_\mu(t)dt - d\mu(t) = \lim_{n\to\infty}\left\{\left[\operatorname{Re}(m/m+1)_u\left(r_n e^{it}\right) - u_{r_n}(t)\right]d\,t\right\}$$

or

$$\operatorname{Re}(m/m+1)_\mu(t)dt - d\mu(t) = \lim_{n\to\infty}\left\{2\operatorname{Re}\left(\frac{1}{V_{m+1}(r^{-1}e^{-it})} - T_\mu\left(\frac{V_{m+1}(x)}{x\,r\,e^{it}-1}\right)\right)d\,t\right\},$$

in the weak-star topology on measures. To complete the *Proof*, write

$$V_{m+1}(x) = \sum_{k=0}^{m} B_k^{(m)} x^k$$

and observe that

$$T_\mu\left(\frac{V_{m+1}(x)}{xre^{it}-1}\right) = -\sum_{\nu=0}^{\infty} r^\nu e^{i\nu t}\sum_{k=0}^{m}\frac{B_k^{(m)}}{2\pi}\int_{-\pi}^{\pi} e^{-i(\nu+k)s}d\,\mu(s)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{\nu=0}^{\infty} r^\nu e^{i\nu(t-s)}\sum_{k=0}^{m} B_k^{(m)} e^{-iks}\right\}d\mu(s)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{V_{m+1}\left(e^{-is}\right)}{re^{i(t-s)}-1}d\,\mu(s),$$

which ends of *Proof*.

Remark 2.1.3. It is of course possible to construct Padé-type approximants to the finite real Baire measure μ on $[-\pi,\pi]$ with various degrees in the numerator and denominator. To do so, let us consider the Fourier-Stieltjes coefficients σ_ν of μ $(\nu = 0,\pm1,\pm2,...)$. The Poisson integral of μ is then defined by

$$u\left(re^{it}\right) = \sum_{\nu=-\infty}^{\infty} \sigma_\nu r^{|\nu|} e^{i\nu t} \quad \left(0 \le r < 1, -\pi \le t \le \pi\right).$$

If, for any $n \ge 0$, T_{μ_n} denotes the linear functional

$$T_{\mu_n} : \mathbb{P}(\mathbb{C}) \to \mathbb{C} : x^\nu \mapsto T_{\mu_n}\left(x^\nu\right) := \sigma_{n+\nu} ,$$

then, as $r \to 1$, the finite real Baire measure

$$\mathrm{Re}(m/m+1)_u\left(re^{it}\right)dt = \left\{2\,\mathrm{Re}\left[\sum_{\nu=0}^{n-1} \sigma_\nu r^\nu e^{i\nu t} + r^n e^{int} T_{\mu_n}\left(G_m(x, re^{it})\right)\right] - \sigma_0\right\} dt$$

has a radial limit

$$\mathrm{Re}(m/m+1)_\mu(t)\,dt = \left\{2\,\mathrm{Re}\left[\sum_{\nu=0}^{n-1} \sigma_\nu e^{i\nu t} + e^{int} T_{\mu_n}\left(G(x, re^{it})\right)\right] - \sigma_0\right\} dt$$

in the weak-star topology on measures. This limit is also called a *Padé-type approximant to* μ . The crucial property is the following one: if $\left|\pi_{m,k}\right| < 1$, for any $k \le m$, the Fourier representation of

$$\mathrm{Re}\left(m+n/m+1\right)_u\left(t\right)dt$$

matches the Fourier series of μ up to the $\pm\left(m+n\right)^{th}$ $-$order's Fourier-Stieltjes term.

Remark 2.1.4. If, instead of a finite real Baire measure, we have to approximate a finite complex Baire measure μ on. $[-\pi,\pi]$, with period 2π $(:\mu(\{-\pi\})=\mu(\{\pi\}))$, of the form

$$d\mu(t)=d\mu^{(1)}(t)+i\,d\mu^{(2)}(t),$$

where $\mu^{(1)}$ and $\mu^{(2)}$ are finite real Baire measures, with Fourier-Stieltjes coefficients

$$\sigma_\nu^{(1)} \text{ and } \sigma_\nu^{(2)} \quad (\nu=0,\pm1,\pm2,...)$$

respectively, then for any $n_1, n_2 \geq 0$, the finite complex Baire measures:

$$\Bigg\{2\,\mathrm{Re}\left[\sum_{\nu=0}^{n_1-1}\sigma_\nu^{(1)}e^{i\nu t}+e^{in_1 t}T_{\mu_{n_1}^{(1)}}\left(Q_m(x,e^{it})\right)\right]$$

$$+i2\,\mathrm{Re}\left[\sum_{\nu=0}^{n_2-1}\sigma_\nu^{(2)}e^{i\nu t}+e^{in_2 t}T_{\mu_{n_2}^{(2)}}\left(Q_m(x,e^{it})\right)\right]-\left(\sigma_0^{(1)}+i\sigma_0^{(2)}\right)\Bigg\}dt\,,$$

$$\Bigg\{2\,\mathrm{Re}\left[\sum_{\nu=0}^{n_1-1}\sigma_\nu^{(1)}e^{i\nu t}+e^{in_1 t}T_{\mu_{n_1}^{(1)}}\left(R_m(x,e^{it})\right)\right]$$

$$+i2\,\mathrm{Re}\left[\sum_{\nu=0}^{n_2-1}\sigma_\nu^{(2)}e^{i\nu t}+e^{in_2 t}T_{\mu_{n_2}^{(2)}}\left(R_m(x,e^{it})\right)\right]-\left(\sigma_0^{(1)}+i\sigma_0^{(2)}\right)\Bigg\}dt\,,$$

$$\Bigg\{2\,\mathrm{Re}\left[\sum_{\nu=0}^{n_1-1}\sigma_\nu^{(1)}e^{i\nu t}+e^{in_1 t}T_{\mu_{n_1}^{(1)}}\left(Q_m(x,e^{it})\right)\right]$$

$$+i2\,\mathrm{Re}\left[\sum_{\nu=0}^{n_2-1}\sigma_\nu^{(2)}e^{i\nu t}+e^{in_2 t}T_{\mu_{n_2}^{(2)}}\left(R_m(x,e^{it})\right)\right]-\left(\sigma_0^{(1)}+i\sigma_0^{(2)}\right)\Bigg\}dt\,,$$

and

$$\left\{2\,\mathrm{Re}\left[\sum_{\nu=0}^{n_1-1}\sigma_\nu^{(1)}e^{i\nu t}+e^{in_1 t}T_{\mu_{n_1}^{(1)}}\left(R_m(x,e^{it})\right)\right]\right.$$

$$\left.+i2\,\mathrm{Re}\left[\sum_{\nu=0}^{n_2-1}\sigma_\nu^{(2)}e^{i\nu t}+e^{in_2 t}T_{\mu_{n_2}^{(2)}}\left(Q_m(x,e^{it})\right)\right]-\left(\sigma_0^{(1)}+i\sigma_0^{(2)}\right)\right\}dt$$

are called *composed Padé-type approximants* to μ. Here,

$$Q_m(x,z) \text{ and } R_m(x,z)$$

denote the interpolation polynomials of $(1-xz)^{-1}$ at (distinct or not) points

$$\pi_{m,0},\pi_{m,1},\dots,\pi_{m,m}\in\mathbb{C} \text{ and } \rho_{m,0},\rho_{m,1},\dots,\rho_{m,m}\in\mathbb{C},$$

respectively. In other words, any finite complex Baire measure of the form:

$$(m/m+1)_\mu dt := \mathrm{Re}(m/m+1)_{\mu^{(1)}}dt+i\,\mathrm{Re}(m/m+1)_{\mu^{(2)}}dt$$

is said to be a *composed Padé-type approximant to*

$$\mu=\mu^{(1)}+i\mu^{(2)}.$$

The error of such an approximation is

$$(m/m+1)_\mu dt-d\mu(t)=\frac{1}{\pi}\left\{\mathrm{Re}\left[\int_{-\pi}^{\pi}\frac{V_{m+1}^{(1)}\left(e^{-is}\right)}{V_{m+1}^{(1)}\left(r^{-1}e^{-it}\right)}\frac{ds}{re^{i(t-s)}-1}\right]\right.$$

$$\left.+i\,\mathrm{Re}\left[\int_{-\pi}^{\pi}\frac{V_{m+1}^{(2)}\left(e^{-is}\right)}{V_{m+1}^{(2)}\left(r^{-1}e^{-it}\right)}\frac{ds}{re^{i(t-s)}-1}\right]\right\},$$

where the limit is considered in the weak-star topology of measures. It is obvious that the computation of composed Padé-type approximants

$$(m/m+1)_{\mu}dt$$

to $d\mu(t)$ requires the knowledge of Fourier-Stieltjes coefficients of

$$d\mu^{(1)} \text{ and } d\mu^{(2)}$$

up to the $\pm m^{th}$ $-$order coefficient.

2.1.2. On the Convergence of a Sequence of Padé-Type Approximants

Given a finite positive Baire measure μ on $[-\pi,\pi]$, we shall now study the assumptions under which a sequence

$$\{R\,\mathrm{e}(m/m+1)_{\mu}(t)dt:\ m=0,1,2,...\}$$

converges to $d\mu(t)$ in the weak-star topology on measures, that is

$$\lim_{m\to\infty}\int_{-\pi}^{\pi} f(t)\ \mathrm{Re}(m/m+1)_{\mu}(t)\ dt = \int_{-\pi}^{\pi} f(t)\ d\mu(t)$$

for any real-valued continuous function $f(t)$ on $[-\pi,\pi]$.

Our first result will follow from a combination of *Radon-Nikodym Theorem* with the error formula given in *Theorem 2.1.2*. Recall that a positive Baire measure μ_1 on $[-\pi,\pi]$ is *absolutely continuous with respect to another positive Baire measure* μ_2 on $[-\pi,\pi]$, if every subset of $[-\pi,\pi]$ of measure zero for μ_2 is a set of measure zero for

μ_1. The *Radon-Nikodym Theorem* states that μ_1 *is absolutely continuous with respect to* μ_2 *if and only if there exists a non-negative function* F *in* $L^1(d\mu_2)$ *satisfying*

$$d\mu_1 = F\, d\mu_2 .$$

We dispose the following partial answer to our problem.

Theorem 2.1.5. *Let* μ *be a finite positive Baire measure on* $[-\pi,\pi]$ *which is absolutely continuous with respect to the Lebesgue measure, and satisfies*

$$\mu(\{-\pi\}) = \mu(\{\pi\}).$$

Suppose there is a constant $K > 0$ *and an open neighborhood* U *of the unit circle into which the generating polynomials* $V_{m+1}(x)$ *satisfy*

$$K \leq \left| V_{m+1}(x) \right| \text{, for any } x \in U \text{ and any } m \text{ enough large.}$$

If the family

$$\{V_{m+1}(\mathrm{e}^{is}) : m = 0,1,2,...\}$$

is an orthonormal bounded system in $L^2[-\pi,\pi]$, *then*

$$\lim_{m\to\infty} \operatorname{Re}(m/m+1)_\mu(t)dt = d\mu(t),$$

in the weak-star topology on measures.

Proof. Let $\varepsilon > 0$ and let $\{r_n < 1 : n = 0,1,2,...\}$ be a strictly increasing sequence of positive numbers such that

$$\lim_{m\to\infty} r_n = 1 \quad \text{and} \quad r_n e^{it} \in U \;\text{ for any }\; n \geq 0 \;\text{ and }\; -\pi \leq t \leq \pi \,.$$

According to the *Radon-Nikodym Theorem*, there is a non-negative function $F \in L^1[-\pi,\pi]$ with

$$d\mu(s) = F(s)\,ds \,.$$

From *Mercer's Theorem*, it then follows that there exists a $M = M(\varepsilon)$ such that the inequalities $m \geq M$ $, n \geq 0$ and $-\pi \leq t \leq \pi$ imply

$$2\left|\int_{-\pi}^{\pi} \frac{1}{r_n e^{i(t-s)} - 1} \frac{V_{m+1}(e^{-is})}{V_{m+1}(r_n^{-1} e^{-it})}\, d\mu(s)\right| = \left|2 \frac{1}{V_{m+1}(r_n^{-1} e^{-it})} \int_{-\pi}^{\pi} \frac{F(s)}{r_n e^{i(t-s)} - 1} V_{m+1}(e^{-is})\, ds\right| < \varepsilon \,.$$

By *Theorem 2.1.2*, we therefore obtain

$$\lim_{m\to\infty} \operatorname{Re}(m/m+1)_\mu(t)dt = d\mu(t),$$

in the weak-star topology on measures, which completes the *Proof.*

Corollary 2.1.6. *Let*

$$V_{m+1}(x) = \sum_{k=0}^{m+1} B_k^{(m)} x^k = \gamma \prod_{k=0}^{m} (x - \pi_{m,k}) \;\; (m = 0,1,2,...)$$

be the generating polynomials of a Padé-type approximation such that

$$\sum_{k=0}^{m+1} \left|B_k^{(m)}\right|^2 = \frac{1}{2\pi} \quad (m \geq 0) \;\; \textit{and} \;\; \sum_{k=0}^{m+1} B_k^{(m)} \overline{B_k^{(n)}} = 0 \quad (m < n) \,.$$

Suppose there are two constants $\sigma < \infty$ *and* $c > 1$ *fulfilling*

$$\sum_{k=0}^{m+1}\left|B_k^{(m)}\right| < \sigma \quad (m \geq 0) \quad and \quad \left|\pi_{m,k}\right| < c \quad (m \geq 0, 0 \leq k \leq m).$$

If μ *is any finite positive Baire measure on* $[-\pi,\pi]$*, that is absolutely continuous with respect to the Lebesgue measure and satisfies* $\mu(\{-\pi\}) = \mu(\{\pi\})$*, then there holds*

$$\lim_{m\to\infty} \mathrm{Re}(m/m+1)_\mu(t)dt = d\mu(t),$$

in the weak-star topology on measures.

Proof. If the generating polynomial

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m}\left(x - \pi_{m,k}\right)$$

is written as

$$V_{m+1}(x) = \sum_{k=0}^{m+1} B_k^{(m)} x^k ,$$

then the orthogonality assumption for the family

$$\left\{V_{m+1}\left(e^{is}\right): \quad m = 0,1,2,\ldots\right\},$$

in the above *Theorem*, is completely described by the following two conditions

$$\sum_{k=0}^{m+1}\left|B_k^{(m)}\right|^2 = \frac{1}{2\pi} \quad (m \geq 0) \ \text{ and } \ \sum_{k=0}^{m+1} B_k^{(m)}\overline{B_k^{(n)}} = 0 \quad (m < n) .$$

In fact, for any $m \geq 0$, we have

$$2\pi\sum_{k=0}^{m+1}\left|B_k^{(m)}\right|^2 = \sum_{k,\nu=0}^{m+1} B_k^{(m)}\overline{B_\nu^{(m)}}\int_{-\pi}^{\pi} e^{iks}e^{-i\nu s}ds = \int_{-\pi}^{\pi}\left(\sum_{k=0}^{m+1} B_k^{(m)}e^{iks}\right)\left(\overline{\sum_{\nu=0}^{m+1} e^{i\nu s}}\right)ds = 1$$

and

$$2\pi\sum_{k=0}^{m+1} B_k^{(m)}\overline{B_k^{(n)}} = \sum_{\nu=0}^{n+1} B_k^{(m)}\overline{B_\nu^{(n)}}\int_{-\pi}^{\pi} e^{iks}e^{-i\nu s}ds = \int_{-\pi}^{\pi}\left(\sum_{k=0}^{m+1} B_k^{(m)}e^{iks}\right)\left(\overline{\sum_{\nu=0}^{n+1} B_\nu^{(n)}e^{i\nu s}}\right)ds = 0\,.$$

Further, the boundedness assumption for the family

$$\left\{V_{m+1}\left(e^{is}\right) : m = 0,1,2,\ldots\right\}$$

is guaranteed by the fact that there is a positive constant $\sigma < \infty$ satisfying

$$\sum_{k=0}^{m+1}\left|B_k^{(m)}\right|^2 < \sigma \quad (m \geq 0)\,.$$

Finally, by the definition of the generating polynomials

$$V_{m+1}(x) = \gamma\prod_{k=0}^{m}\left(x - \pi_{m,k}\right) \quad \left(\gamma \in \mathbb{C} - \{0\}\right),$$

it is easily seen that the existence of a constant $c > 1$ satisfying

$$\left|\pi_{m,k}\right| \leq c \quad \text{for any } m \text{ and } k$$

carries along the existence of an open neighborhood U of the unit circle C into which there holds

$$0 < K \leq \inf_{z\in\overline{U}}\left|V_{m+1}(z)\right|, \quad (m = 0,1,2,\ldots)$$

for some positive constant K which is independent of m.

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Theorem 2.1.5 can be viewed as an analogous to *Theorem 1.3.17*. A natural question which now arises is whether *Theorem 1.3.23* can be extended to the case of a finite Baire measure on $[-\pi, \pi]$. This question has an affirmative answer, as we shall now see.

Theorem 2.1.7. *Suppose the polynomials*

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} \left(x - \pi_{m,k}\right)$$

fulfil

$$\lim_{m \to \infty} \frac{V_{m+1}(x)}{V_{m+1}\left(z^{-1}\right)} = 0$$

compactly in an open subset ω *of* $\mathbb{C}^2$ *containing* $(\mathbb{C} \times \{0\}) \cup \left(\overline{D} \times D\right)$. *Let* μ *be any finite real Baire measure on* $[-\pi, \pi]$, *with period* 2π. *If*

$$\left\{ \mathrm{Re}(m/m+1)_{\mu}(t)\, dt : m = 0,1,2,\ldots \right\}$$

is the Padé-type approximation sequence to μ, *with generating sequence*

$$\left\{ V_{m+1}(x) : m = 0,1,2,\ldots \right\},$$

then

$$\lim_{m \to \infty} \mathrm{Re}(m/m+1)_{\mu}(t) dt = d\mu(t),$$

in the weak-star topology on measures.

Proof. Let f be a real-valued continuous function defined on $[-\pi,\pi]$. Since the Poisson integral $u(z)=u\left(re^{it}\right)$ of μ is harmonic in the disk, *Theorem 1.2.12* can be applied to get

$$\lim_{m\to\infty}\operatorname{Re}(m/m+1)_u(z)=u(z),$$

compactly in

$$\mathrm{g}(\omega)=\{z\in D:(\zeta,z)\in\omega,|\zeta|\leq 1\}.$$

Since $\overline{D}\times D\subset\omega$, we have $\mathrm{g}(\omega)=D$. This means that

$$\lim_{m\to\infty}\left[f(t)\operatorname{Re}(m/m+1)_u\left(re^{it}\right)\right]=f(t)u_r(t)\quad\left(u_r(t)=u\left(re^{it}\right)\right),$$

uniformly on $[-\pi,\pi]$, for any fixed $r<1$. From *Lebesgue's Dominated Convergence Theorem* and from the fact that the measures

$$d\mu_r(t)=u_r(t)\,dt$$

converge to $d\mu(t)$ in the weak-star topology on measures, it follows that

$$\lim_{r\to 1}\lim_{m\to\infty}\int_{-\pi}^{\pi}f(t)\operatorname{Re}(m/m+1)_u\left(re^{it}\right)dt=\lim_{r\to 1}\int_{-\pi}^{\pi}f(t)u_r(t)\,dt$$

$$=\lim_{r\to 1}\int_{-\pi}^{\pi}f(t)\,d\mu_r(t)dt=\int_{-\pi}^{\pi}f(t)\,d\mu(t).$$

Now, recall that the sequence

$$\{\operatorname{Re}(m/m+1)_\mu(t)dt:m=0,1,2,...\}$$

with generating polynomials $\{V_{\mathrm{m}+1}(x):\ m=0,1,2,...\}$ converges to $d\mu(t)$ in the weak-star topology on measures if and only if

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} f(t)\operatorname{Re}(m/m+1)_{\mu}(t)\,dt = \int_{-\pi}^{\pi} f(t)\,d\mu(t),$$

or equivalently if and only if

$$\lim_{m\to\infty} \lim_{r\to 1} \int_{-\pi}^{\pi} f(t)\operatorname{Re}(m/m+1)_{u}\left(re^{it}\right)dt$$

$$= \lim_{r\to 1} \int_{-\pi}^{\pi} f(t)u_r(t)\,dt = \lim_{r\to 1} \int_{-\pi}^{\pi} f(t)\,d\mu(t),$$

for any f. Thus, to prove the *Theorem*, it is enough to show that

$$\lim_{m\to\infty} \lim_{r\to 1} \int_{-\pi}^{\pi} f(t)\operatorname{Re}(m/m+1)_{u}\left(re^{i(t+\theta)}\right)dt$$

$$= \lim_{r\to 1} \lim_{m\to\infty} \int_{-\pi}^{\pi} f(t)\operatorname{Re}(m/m+1)_{u}\left(re^{i(t+\theta)}\right)dt,$$

for any $\theta \in [-\pi,\pi]$.

Denote by

$$\sum_{\nu=-\infty}^{\infty} d_{\nu}^{(m)} r^{|\nu|} e^{i\nu(t+\theta)}$$

the Fourier representation of

$$\operatorname{Re}(m/m+1)_{u}(r.e^{i\nu(t+\theta)}).$$

Since $\operatorname{Re}(m/m+1)_{u}(z)$ is harmonic in z, it is twice continuously differentiable and therefore, the sequence of partial sums

$$\left\{ \sum_{\nu=-n}^{n} d_{\nu}^{(m)} r^{|\nu|} e^{i\nu(t+\theta)} : n = 0,1,2,... \right\}$$

converges uniformly on $[-\pi, \pi]$ to

$$\mathrm{Re}(m/m+1)_u (r.e^{i(t+\theta)}),$$

because two integrations by parts show that $d_{\nu}^{(m)}\, r^{|\nu|} = \mathrm{O}(\nu^{-2})$. It follows that it suffices to prove that

$$\lim_{m\to\infty} \lim_{r\to 1} \sum_{\nu=-\infty}^{\infty} d_{\nu}^{(m)} \left(\int_{-\pi}^{\pi} f(t) e^{i\nu t}\, dt \right) r^{|\nu|} e^{i\nu\theta}$$

$$= \lim_{r\to 1} \lim_{m\to\infty} \sum_{\nu=-\infty}^{\infty} d_{\nu}^{(m)} \left(\int_{-\pi}^{\pi} f(t) e^{i\nu t} dt \right) r^{|\nu|} e^{i\nu\theta},$$

for all $\theta \in [-\pi, \pi]$. But

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{i\nu t} dt$$

is the Fourier coefficient $\tau_{-\nu}^{(f)}$ of f. Thus, it suffices to show that

$$\lim_{m\to\infty} \lim_{r\to 1} \sum_{\nu=-\infty}^{\infty} d_{\nu}^{(m)}\, \tau_{-\nu}^{(f)}\, r^{|\nu|}\, e^{i\nu\theta} = \lim_{r\to 1} \lim_{m\to\infty} \sum_{\nu=-\infty}^{\infty} d_{\nu}^{(m)}\, \tau_{-\nu}^{(f)}\, r^{|\nu|}\, e^{i\nu\theta},$$

for any $\theta \in [-\pi, \pi]$.

Notice that the sequence

$$\left\{ d_{\nu}^{(m)}\, \tau_{-\nu}^{(f)} \,:\, \nu = 0, \pm 1, \pm 2, ... \right\}$$

is the sequence of Fourier coefficients of a continuous function in $L^2[-\pi,\pi]$. In fact, by *Cauchy-Schwarz's Inequality*, we have

$$\sum_{\nu=-\infty}^{\infty}\left|d_\nu^{(m)}\tau_{-\nu}^{(f)}\right|^2 \le \left(\sum_{\nu=-\infty}^{\infty}\left|d_\nu^{(m)}\tau_{-\nu}^{(f)}\right|\right)^2 \le \left(\sum_{\nu=-\infty}^{\infty}\left|d_\nu^{(m)}\right|^2\right)\left(\sum_{\nu=-\infty}^{\infty}\left|\tau_{-\nu}^{(f)}\right|^2\right).$$

From *Bessel's Inequalities*

$$\left(\sum_{\nu=-\infty}^{\infty}\left|d_\nu^{(m)}\right|^2\right) \le \left\|\mathrm{Re}(m/m+1)_u(e^{it})\right\|_2 < \infty \quad \text{and} \quad \sum_{\nu=-\infty}^{\infty}\left|\tau_{-\nu}^{(f)}\right|^2 \le \|f\|_2 < \infty,$$

it follows that

$$\sum_{\nu=-\infty}^{\infty}\left|d_\nu^{(m)}\tau_{-\nu}^{(f)}\right|^2 < \infty .$$

Application of the *Riesz-Fisher Theorem* guarantees now that the sequence

$$\left\{d_\nu^{(m)}\tau_{-\nu}^{(f)} \ : \ \nu = 0,\pm1,\pm2,\ldots\right\}$$

is, in fact, the sequence of Fourier coefficients of a continuous L^2-function. Let $H_m(\theta)$ be this function, with Fourier coefficients

$$b_\nu^{(\mathrm{m})} := d_\nu^{(m)}\tau_{-\nu}^{(f)} \ : \ (\nu = 0,\pm1,\pm2,\ldots).$$

Extend $H_m(\theta)$ into the unit disk by defining its Poisson integral

$$H_m\left(re^{i\theta}\right) = \sum_{\nu=-\infty}^{\infty} b_\nu^{(m)} r^{|\nu|} e^{i\nu\theta} \quad (0 \le r < 1) .$$

With this notation, to prove the *Theorem* it is enough to show that

$$\lim_{m\to\infty} H_m(\theta) = \lim_{r\to1}\lim_{m\to\infty} H_m\left(re^{i\theta}\right),$$

where the limits $\lim_{m\to\infty}$ are considered with respect to the $L^{\infty}-$norm in $[-\pi,\pi]$. In other words, it is sufficient to show that there exists a function $H^*(\theta)\in C[-\pi,\pi]$ with the following two properties:

$$\text{(P1)}\quad \lim_{r\to 1}\lim_{m\to\infty}\left\|H_m\left(re^{i\theta}\right)-H^*(\theta)\right\|_{\infty}=0\,,$$
$$\text{(P2)}\quad \lim_{m\to\infty}\left\|H_m(\theta)-H^*(\theta)\right\|_{\infty}=0\,.$$

To do so, we may make use of three observations:

(i). First, it is known that, for each fixed m, the functions $H_m\left(re^{i\theta}\right)$ converge to $H_m(\theta)$ in $L^{\infty}-$norm, i.e.

$$\lim_{r\to 1}\left\|H_m\left(re^{i\theta}\right)-H_m(\theta)\right\|_{\infty}=0\,.$$

(ii). Next, for each fixed $r<1$, the sequence $\{H_m\left(re^{i\theta}\right)$: m=0,1,2,...} converges to some $H^*(\theta)\in[-\pi,\pi]$, i.e.

$$\lim_{m\to\infty}\left\|H_m\left(re^{i\theta}\right)-H^*(\theta)\right\|_{\infty}=0\,.$$

(To see this, one may use *Cauchy-Schwarz' Inequality* to obtain

$$\left|H_m\left(re^{i\theta}\right)-H_n\left(re^{i\theta}\right)\right|\le\left(\sum_{\nu=-\infty}^{\infty}\left|d_{\nu}^{(m)}r^{|\nu|}-d_{\nu}^{(n)}r^{|\nu|}\right|^2\right)^{\frac{1}{2}}\left(\sum_{\nu=-\infty}^{\infty}\left|\tau_{-\nu}^{(f)}\right|^2\right)^{\frac{1}{2}}.$$

Then, by *Bessel's Inequality*, there is a positive constant c_f such that

$$\left\|H_m\left(re^{i\theta}\right)-H_n\left(re^{i\theta}\right)\right\|_{\infty}\le c_f\left\|\operatorname{Re}(m/m+1)_u\left(re^{i\theta}\right)-\operatorname{Re}(n/n+1)_u\left(re^{i\theta}\right)\right\|_2$$
$$\le(2\pi)c_f\left\|\operatorname{Re}(m/m+1)_u\left(re^{i\theta}\right)-\operatorname{Re}(n/n+1)_u\left(re^{i\theta}\right)\right\|_{\infty}.$$

Since u is harmonic in the disk, an application of *Theorem 1.2.12* shows that the sequence $\left\{H_m\left(re^{i\theta}\right): m=0,1,2,\ldots\right\}$ is a Cauchy sequence into the complete space $C[-\pi,\pi]$, which proves our assertion.)

(iii). Finally, if $H^*\left(r e^{i\theta}\right)$ is the Poisson integral of the continuous function $H^*(\theta)$, then

$$\lim_{r\to 1}\left\|H^*\left(re^{i\theta}\right)-H^*(\theta)\right\|_{\infty}=0.$$

After these remarks, one can really end the *Proof* of the *Theorem*. From (ii), it follows directly the asymptotic formula

$$\lim_{r\to 1}\lim_{m\to\infty}\left\|H_m\left(re^{i\theta}\right)-H^*(\theta)\right\|_{\infty}=0.$$

On the other hand, since, by (i) and (iii), for any $\varepsilon>0$ there is a $\tilde{r}=\tilde{r}(\varepsilon)<1$ with

$$\left\|H_m(\theta)-H_m\left(re^{i\theta}\right)\right\|_{\infty}<\frac{\varepsilon}{3} \quad \text{and} \quad \left\|H^*\left(re^{i\theta}\right)-H^*(\theta)\right\|_{\infty}<\frac{\varepsilon}{3}$$

for every $r\geq\tilde{r}$, we obtain

$$\begin{aligned}&\left\|H^*(\theta)-H_m(\theta)\right\|_{\infty}\\ &\qquad\leq\left\|H^*(\theta)-H^*\left(re^{i\theta}\right)\right\|_{\infty}+\left\|H_m(\theta)-H_m\left(re^{i\theta}\right)\right\|_{\infty}+\left\|H^*\left(re^{i\theta}\right)-H_m\left(re^{i\theta}\right)\right\|_{\infty}\\ &<\frac{2\varepsilon}{3}\left\|H^*\left(re^{i\theta}\right)-H_m\left(re^{i\theta}\right)\right\|_{\infty}.\end{aligned}$$

Obviously, one can find an integer $\tilde{M}=\tilde{M}\left(\tilde{r}\right)>0$ such that for every $m\geq\tilde{M}$,

$$\left\|H^*\left(re^{i\theta}\right)-H_m\left(re^{i\theta}\right)\right\|_{\infty}<\frac{\varepsilon}{3}.$$

We thus conclude that

$$\lim_{m\to\infty}\left\|H_m(\theta)-H^*(\theta)\right\|_{\infty}=0,$$

Hence the function $H^*(\theta)\in C[-\pi,\pi]$ has the properties (P1) and (P2). This completes the *Proof*.

As we have seen in *Theorems 1.3.23* and *2.1.7*, the crucial hypothesis for the convergence of a Padé-type approximation sequence to a continuous function or to a finite Baire measure concerns the choice of the generating polynomials

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} \left(x - \pi_{m,k}\right)$$

or equivalently the choice of the interpolation points $\pi_{m,k}$. In both cases, remind that the sufficient condition was the compact convergence of the sequence

$$\left\{ \frac{V_{m+1}(x)}{V_{m+1}\left(z^{-1}\right)} \ : \ m = 0,1,2,.. \right\}$$

into an open subset of $\mathbb{C}^2$ containing $(\mathbb{C}\times\{0\}) \cup \left(\overline{D} \times D\right)$. Our next purpose is to give a stronger sufficient convergence condition in terms of the entries $\pi_{m,k}$ only.

Corollary 2.1.8. *Suppose the interpolation points* $\pi_{m,k}$ *(*$m \geq 0$ *,* $0 \leq k \leq m$ *) are chosen so that*

$$-1 < \pi_{m,k} < 1 \quad and \quad \lim_{m\to\infty} \sum_{n\geq 1} \frac{1}{n} \sum_{k=0}^{n} \left(\pi_{m,k}\right)^2 = -\infty \, ..$$

(a). *For any real-valued continuous* $2\pi -$*periodic function* f *defined on* $[-\pi,\pi]$*, there holds*

$$\lim_{m\to\infty} R\mathrm{e}\left(m/m+1\right)_f(t) = f(t)$$

point-wise on $[-\pi,\pi]$.

(b). *For any finite real Baire measure* μ *on* $[-\pi,\pi]$*, satisfying*

$$\mu(\{-\pi\}) = \mu(\{\pi\}),$$

there holds

$$\lim_{m\to\infty} R\mathrm{e}\left(m/m+1\right)_\mu(t)dt = d\mu(t)$$

in the weak-star topology on measures.

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Proof. According to *Theorems 1.3.23* and *2.1.7*, it is enough to show that

$$\lim_{m\to\infty}\frac{V_{m+1}(x)}{V_{m+1}(z^{-1})}=\lim_{m\to\infty}\frac{\prod_{k=0}^{m}(x-\pi_{m,k})}{\prod_{k=0}^{m}(z^{-1}-\pi_{m,k})}=0\,,$$

compactly in an open neighborhood of $(\mathbb{C}\times\{0\})\cup(\overline{D}\times D)$.

First, we shall prove that

$$\lim_{m\to\infty}\frac{\prod_{k=0}^{m}(x-\pi_{m,k})}{\prod_{k=0}^{m}(z^{-1}-\pi_{m,k})}=0\,,$$

compactly into an open neighborhood of $\overline{D}\times D$.

Let $\varepsilon>0, 0\le r<1$ and $0<\delta<1$. It is clear that our hypothesis is equivalent to the limit condition:

$$\lim_{m\to\infty}\exp\left\{\sum_{n\ge1}-\frac{3}{n}\sum_{k=0}^{m}(\pi_{m,k})^{n}\right\}=\infty\,.$$

If follows that, for any $x\in\mathbb{C}$ with $|x|=1+\delta$ and any $z\in D$ with $|z|\le r$, we have

$$\lim_{m\to\infty}\prod_{n\ge1}\left|\exp\left\{\frac{\sum_{k=0}^{m}(\pi_{m,k})^{n}}{n}\left(\frac{1}{x^{n}}-z^{n}\right)\right\}\right|=\lim_{m\to\infty}\prod_{n\ge1}\exp\left\{\frac{\sum_{k=0}^{m}(\pi_{m,k})^{n}}{n}\mathrm{Re}\left(\frac{1}{x^{n}}-z^{n}\right)\right\}$$

$$\ge\lim_{m\to\infty}\prod_{n\ge1}\exp\left\{\frac{\sum_{k=0}^{m}(\pi_{m,k})^{n}}{n}\left[-\frac{1}{1+\delta}-r\right]\right\}$$

$$\ge\lim_{m\to\infty}\prod_{n\ge1}\exp\left\{-\frac{3}{n}\sum_{k=0}^{m}(\pi_{m,k})^{n}\right\}$$

$$= \infty .$$

This means that there exists an integer $M_1 = M_1(\varepsilon) \geq 0$, with

$$\frac{1}{\varepsilon} < \prod_{k=0}^{m} \prod_{n \geq 1} \left| \exp\left\{ \frac{(\pi_{m,k})^n}{nx^n} - \frac{(\pi_{m,k})^n z^n}{n} \right\} \right|,$$

for any $m \geq M_1$ .In other words, there exists a $M_1 = M_1(\varepsilon)$ such that $m \geq M_1$ and $|z| \leq r$ implies

$$\varepsilon > \exp\left\{ \mathrm{Re} \sum_{k=0}^{m} \sum_{n \geq 1} \frac{1}{n} (\pi_{m,k})^n \left[z^n - x^{-n} \right] \right\},$$

or, in other words,

$$\sum_{k=0}^{m} \mathrm{Re}\left\{ \log 1 - \sum_{n \geq 1} \frac{1}{n} \left(\frac{\pi_{m,k}}{x} \right)^n \right\} < \log \varepsilon + \sum_{k=0}^{m} \mathrm{Re}\left\{ \log 1 - \sum_{n \geq 1} \frac{1}{n} (z\,\pi_{m,k})^n \right\}.$$

Observe that the expressions

$$\mathrm{Re}\left\{ \log 1 - \sum_{n \geq 1} \frac{1}{n} \left(\frac{\pi_{m,k}}{x} \right)^n \right\} \quad \text{and} \quad \mathrm{Re}\left\{ \log 1 - \sum_{n \geq 1} \frac{1}{n} (z\,\pi_{m,k})^n \right\}$$

are the Taylor series developments of the functions

$$\log \left| 1 - \frac{\pi_{m,k}}{x} \right| \quad \text{and} \quad \log \left| 1 - \pi_{m,k} z \right|,$$

respectively. We can therefore rewrite our last inequality as

$$\sum_{k=0}^{m} \log \left| \frac{x - \pi_{m,k}}{x} \right| < \log \varepsilon + \sum_{k=0}^{m} \log \left| 1 - z\,\pi_{m,k} \right|$$

for $m \geq M_1$, $|x| = 1 + \delta, |z| \leq r$. By the maximum principle for subharmonic functions, we immediately obtain

$$\sum_{k=0}^{m} \log \left| x - \pi_{m,k} \right| < \log \varepsilon + \sum_{k=0}^{m} \log \left| \frac{1}{z} - \pi_{m,k} \right| + \log \left[r^{m+1} (1+\delta)^{m+1} \right]$$

for $m \geq M_1$, $|x| \leq 1+\delta$, $|z| \leq r$. This can also be written in the form

$$\prod_{k=0}^{m}\left|x-\pi_{m,k}\right| < \varepsilon \prod_{k=0}^{m}\left|\frac{1}{z}-\pi_{m,k}\right| r^{m+1}\left(1+\delta\right)^{m+1} \quad (m \geq M_1, |x| \leq 1+\delta, |z| \leq r).$$

If, in particular,

$$\delta = \delta_r = \frac{1-r}{r} \quad (\Leftrightarrow (1-\delta_r)r = 1),$$

then we get

$$\lim_{m\to\infty} \frac{\prod_{k=0}^{m}\left(x-\pi_{m,k}\right)}{\prod_{k=0}^{m}\left(z^{-1}-\pi_{m,k}\right)} = 0$$

uniformly on

$$\left\{(x,z) \in \mathbb{C}^2 : |x| \leq \frac{1}{r}, |z| \leq r\right\}.$$

Summarizing, we have showed that

$$\lim_{m\to\infty} \frac{\prod_{k=0}^{m}\left(x-\pi_{m,k}\right)}{\prod_{k=0}^{m}\left(z^{-1}-\pi_{m,k}\right)} = 0$$

compactly into the open neighborhood

$$\varpi := \bigcup_{0\leq r<1}\left\{(x,z) \in \mathbb{C}^2 : |x| \leq \frac{1}{r}, |z| \leq r\right\} \quad \text{of } \overline{\mathrm{D}} \times D.$$

Since ϖ contains also $\mathbb{C} \times \{0\}$, the *Proof* of *Corollary 2.1.8* is complete.

2.2. Integral Representations

2.2.1. Integral Representations and Padé-Type Operators

Until now, we have defined and studied Padé and Padé-type approximation to harmonic functions in the unit disk *D*, as well as to $2\pi-$periodic real-valued L^p-functions on the unit circle C or the compact interval $[-\pi,\pi]$. In any case, the structural development and main ideas of our theory were analogous to the classical theory on rational approximation to analytic functions.

Really, no situation is quite as pleasant as the L^2-case. In this *Paragraph*, we shall look for another way to introduce Padé-type approximants to L^2-functions and to harmonic functions. Our method will rely on integral representation formulas and lead to a number of interesting approximation results.

To begin our discussion, let us consider any real-valued L^2-function $u(z)$ defined on the circle C. Suppose the Fourier series expansion of $u(e^{it})$ is

$$\sum_{v=-\infty}^{\infty}\sigma_v e^{ivt}.$$

Since u is square integrable, the sequence of its partial sums

$$\left\{\sum_{v=-n}^{n}\sigma_v e^{ivt} : n=0,1,2,...\right\}$$

converges to $u(e^{it})$ in the L^2-norm. Let $\mathbb{P}(\mathbb{C})$ be the vector space of all complex-valued analytic polynomials with coefficients in $\mathbb{C}$. For every

$$p(x)=\sum_{v=0}^{m}\beta_v x^v \in \mathbb{P}(\mathbb{C}),$$

we denote by $\overline{p}(x)$ the polynomial

$$\overline{p}(x)=\sum_{\nu=0}^{m}\overline{\beta}_{\nu}x^{\nu}\in \mathbb{P}(\mathbb{C}).$$

Define the linear functionals

$$T_u:\mathbb{P}(\mathbb{C})\rightarrow\mathbb{C} \text{ and } S_u:\mathbb{P}(\mathbb{C})\rightarrow\mathbb{C}$$

associated with u by

$$T_u\left(x^{\nu}\right):=\sigma_{\nu} \quad and \quad S_u\left(x^{\nu}\right):=\sigma_{-\nu} \qquad (\nu=0,1,2,...).$$

As it is well known, the Poisson integral of $u(z)=u\left(e^{it}\right)$ $\left(|z|=1\right)$ extends to a harmonic real-valued function

$$u(z)=u\left(re^{it}\right)$$

in the unit disk D $\left(|z|<1, 0\leq r<1\right)$. This harmonic function being the real part of some analytic function in D, we immediately see that

$$\overline{T_u\left(x^{\nu}\right)}=\overline{\sigma}_{\nu}=\sigma_{-\nu}=S_u\left(x^{\nu}\right)$$

for any $\nu\geq 0$.

More generally, we have the following

Proposition 2.2.1. *For every* $p(x)\in\mathbb{P}(\mathbb{C})$, *it holds*

$$\overline{S_u(p(x))}=T_u(\overline{p}(x)) \quad and \quad \overline{S_u(\overline{p}(x))}=T_u(p(x)).$$

Proof. Let

$$p(x)=\sum_{\nu=0}^{m}\beta_{\nu}x^{\nu}\in \mathbb{P}(\mathbb{C}).$$

By linearity, we obtain

$$S_u(p(x)) = S_u\left(\sum_{v=0}^{m} \beta_v x^v\right) = \sum_{v=0}^{m} \beta_v S_u(x^v)$$

$$= \sum_{v=0}^{m} \beta_v \overline{T_u(x^v)} = \overline{\sum_{v=0}^{m} \overline{\beta}_v T_u(x^v)}$$

$$= \overline{T_u\left(\sum_{v=0}^{m} \overline{\beta}_v x^v\right)} = \overline{T_u(\overline{p}(x))},$$

and moreover

$$\overline{S_u(\overline{p}(x))} = \overline{S_u\left(\sum_{v=0}^{m} \overline{\beta}_v x^v\right)} = \overline{\sum_{v=0}^{m} \overline{\beta}_v S_u(x^v)}$$

$$= \overline{\sum_{v=0}^{m} \overline{\beta}_v \overline{T_u(x^v)}} = \sum_{v=0}^{m} \beta_v T_u(x^v)$$

$$= T_u\left(\sum_{v=0}^{m} \beta_v x^v\right) = T_u(p(x)).$$

Corollary 2.2.2. *For every* $p(x) \in \mathbb{P}(\mathbb{C})$, *there holds*

$$\operatorname{Re} T_u(\overline{p}(x)) = \operatorname{Re} S_u(p(x))$$

and

$$\operatorname{Re} T_u(p(x)) = \operatorname{Re} S_u(\overline{p}(x)).$$

Now, observe that the linear functional S_u can be extended continuously on the space $L^2(C)$ of all complex-valued that are square integrable functions on the unit circle C. Indeed, if

$$p(x)=\sum_{v=0}^{m} \beta_v x^v \in \mathbb{P}(\mathbb{C}),$$

then, by *Hölder's Inequality*, we get

$$\left|S_u(p(x))\right|^2=\left|\sum_{v=0}^{m} \beta_v \sigma_{-v}\right|^2=\left|\sum_{v=0}^{m} \overline{\beta}_v \sigma_v\right|^2$$

$$=\left|\frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} u(e^{it})\left(\sum_{v=0}^{m} \overline{\beta}_v \cdot e^{-ivt}\right)dt\right|^2$$

$$=\left|\frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} u(e^{it})\,\overline{p(e^{it})}dt\right|^2 \leq c_u \left\|p(x)\right\|_2^2,$$

for some positive constant c_u depending only on u. Hence, by the *Hahn-Banach Theorem*, there is a continuous linear extension of S_u on $L^2(C)$. It follows, from the *Riesz Representation Theorem*, that there exists a unique $F_u \in L^2(C)$ such that

$$S_u(g)=\int_C g(\zeta)\,\overline{F_u(\zeta)}d\zeta=i\int_{-\pi}^{\pi} g(e^{i\theta})\,\overline{F_u(e^{i\theta})}\,e^{i\theta}d\theta$$

for all $g \in L^2(C)$. If, in particular,

$$g(\zeta)=\zeta^v,$$

then

$$S_u(\zeta^v)=\int_C \zeta^v\,\overline{F_u(\zeta)}d\zeta=i\int_{-\pi}^{\pi} e^{iv\theta}\,\overline{F_u(e^{i\theta})}\,e^{i\theta}d\theta.$$

But

$$S_u(\zeta^v)=\sigma_{-v}=\frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{i\theta})\,e^{iv\theta}d\theta,$$

and therefore

$$\overline{F_u(e^{i\theta})}=-i\,u(e^{i\theta})\,e^{-i\theta}.$$

This implies that

$$S_u(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\, u(e^{i\theta})\, d\theta$$

for all $g \in L^2(C)$. In view of *Corollary 2.2.2*, we have thus obtained the

Theorem 2.2.3. *Let*

$$\mathrm{M} = (\pi_{m,k})_{m \ge 0, 0 \le k \le m}$$

be an infinite triangular interpolation matrix with complex entries and, for any $m \ge 0$, *let* $G_m(x,z)$ *be the unique polynomial of degree at most* m *which interpolates the function*

$$(1 - xz)^{-1}$$

at $x = \pi_{m,0}, \pi_{m,1}, \pi_{m,2}, \ldots, \pi_{m,m}$ (z *: fixed and* $|\pi_{m,k}| < 1$).

(a). *For any real-valued function* $u \in L^2(C)$, *the corresponding Padé-type approximant* $\mathrm{Re}(m/m+1)_u(z)$ *to* $u(z)$ *has the following integral representation*

$$\mathrm{Re}(m/m+1)_u(z) = \frac{1}{2\pi i} \cdot \int_C u(\zeta)\, \frac{\mathrm{Re}\{4\pi \overline{G}_m(\zeta, z) - 1\}}{\zeta}\, d\zeta \qquad (|z| = 1).$$

Equivalently,

$$\begin{aligned}
\mathrm{Re}(m/m+1)_u(e^{it}) &= \int_{-\pi}^{\pi} u(e^{i\theta})\, 2\,\mathrm{Re}\left\{\overline{G}_m(e^{i\theta}, e^{it}) - \frac{1}{4\pi}\right\} d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta})\, \mathrm{Re}\{4\pi \cdot \overline{G}_m(e^{i\theta}, e^{it}) - 1\}\, d\theta \quad (-\pi \le t \le \pi).
\end{aligned}$$

(b). *Let* $f \in L^2[-\pi, \pi]$ *be a* $2\pi-$*periodic real-valued function, with Fourier coefficients* $\{c_v : v = 0, \pm 1, \pm 2, \ldots\}$. *Since*

$$f(t) = \sum_{\nu=-\infty}^{\infty} c_\nu \cdot e^{i\nu t}$$

in the L^2-*norm, the function* $f(t)$ *can be viewed as a function of the unit circle, and therefore the Padé-type approximant* $\mathrm{Re}(m/m+1)_f(t)$ *to* $f(t)$ *has the following integral representation*

$$\mathrm{Re}(m/m+1)_f(t) = \int_{-\pi}^{\pi} f(\theta)\, 2\,\mathrm{Re}\left\{\overline{G}_m\left(e^{i\theta}, e^{it}\right) - \frac{1}{4\pi}\right\} d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,\mathrm{Re}\left\{4\pi\cdot\overline{G}_m\left(e^{i\theta}, e^{it}\right) - 1\right\} d\theta \quad (-\pi \le t \le \pi).$$

In order to simplify the formalism, we shall also make use of the notation

$$\frac{\mathrm{Re}\, B_m(\zeta, z)}{\zeta}$$

for the kernel

$$\frac{\mathrm{Re}\{4\pi\, \overline{G}_m(\zeta, z) - 1\}}{\zeta},$$

i.e.

$$\frac{\mathrm{Re}\, B_m(\zeta, z)}{\zeta} := \frac{\mathrm{Re}\{4\pi\, \overline{G}_m(\zeta, z) - 1\}}{\zeta}$$

and

$$\mathrm{Re}\, B_m\left(e^{i\theta}, e^{it}\right) := \mathrm{Re}\left\{4\pi\, \overline{G}_m\left(e^{i\theta}, e^{it}\right) - 1\right\}.$$

As it is pointed out in *Paragraph 1.3.2*, the function $\mathrm{Re}(m/m+1)_u(z)$ $(|z| = 1)$ is continuous. Hence, the integral operator $\mathrm{Re}(m/m+1)$ maps $L^2_{\mathbb{R}}(C)$ into $L^2_{\mathbb{R}}(C)$ and therefore, by the *Closed Graph Theorem*, it is continuous (of course, under the assumption that $|\pi_{m,k}| < 1$ for all $k \le m$). Here, $L^2_{\mathbb{R}}(C)$ denotes the Hilbert space of all real-valued functions that are square integrable in the circle C. The integral operator

$$\mathrm{Re}(m/m+1): L^2_{\mathbb{R}}(C) \to L^2_{\mathbb{R}}(C):$$

$$u(z) \mapsto \mathrm{Re}(m/m+1)_u(z) = \frac{1}{2\pi i}\int_C u(\zeta)\frac{\mathrm{Re}\, B_m(\zeta,z)}{\zeta}\, d\zeta$$

is called the *Padé-type operator* for $L^2_{\mathbb{R}}(C)$.

Its adjoint is given by

$$\mathrm{Re}(m/m+1)^* : L^2_{\mathbb{R}}(C) \to L^2_{\mathbb{R}}(C):$$

$$u(z) \mapsto \mathrm{Re}(m/m+1)^*_u(z) = \frac{1}{2\pi i}\int_C u(\zeta)\frac{\mathrm{Re}\, B_m(z,\zeta)}{z}\, d\zeta\,.$$

In fact, to $\mathrm{Re}(m/m+1)$ there corresponds a unique operator

$$\mathrm{Re}(m/m+1)^* : L^2_{\mathbb{R}}(C) \to L^2_{\mathbb{R}}(C)$$

satisfying

$$\left\langle \mathrm{Re}(m/m+1)_u, w\right\rangle = \left\langle u, \mathrm{Re}(m/m+1)^*_w\right\rangle,$$

i.e.

$$\int_C \mathrm{Re}(m/m+1)_u(\zeta)\, w(\zeta)\, d\zeta = \int_C u(z)\, \mathrm{Re}(m/m+1)^*_w(z)\, dz$$

for all $u, w \in L^2(C)$. Since, by *Fubini's Theorem*,

$$\int_C \mathrm{Re}(m/m+1)_u(\zeta)\, w(\zeta) d\zeta = \int_C \frac{1}{2\pi i}\int_C u(z)\frac{\mathrm{Re}\, B_m(z,\zeta)}{z}\, dz \;\; w(\zeta) d\zeta$$

$$= \int_C u(z)\left(\frac{1}{2\pi i}\int_C w(\zeta)\, \frac{\mathrm{Re}\, B_m(z,\zeta)}{z}\, d\zeta\right) dz\,,$$

we conclude that

$$\mathrm{Re}(m/m+1)_w^*(z) = \frac{1}{2\pi i}\int_C w(\zeta)\,\frac{\mathrm{Re}\,B_m(z,\zeta)}{z}\,d\zeta \qquad \left(w \in L^2_{\mathbb{R}}(C)\right).$$

Similarly, as it is pointed out in *Paragraph 1.3.2*, for any real-valued $2\pi-$periodic function $f \in L^2[-\pi,\pi]$, the Padé-type approximant $\mathrm{Re}(m/m+1)_f(t)$ is continuous, and, by construction, $2\pi-$periodic. It follows that the integral operator $\mathrm{Re}(m/m+1)$ maps the space $L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$ of real-valued $2\pi-$periodic functions of $L^2[-\pi,\pi]$ into itself. Hence, by the *Closed Graph Theorem*, this operator

$$\mathrm{Re}(m/m+1): L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi] \to L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]:$$

$$f(t) \mapsto \mathrm{Re}(m/m+1)_f(t) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(\theta)\;\mathrm{Re}\,B_m\left(e^{i\theta},e^{it}\right)d\theta$$

is continuous. It is called the *Padé-type operator* for $L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$. Its adjoint operator is then given by

$$\mathrm{Re}(m/m+1)^* : L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi] \to L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$$

$$: f(t) \mapsto \mathrm{Re}(m/m+1)_f^*(t) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} f(\theta)\;\mathrm{Re}\,B_m\left(e^{it},e^{i\theta}\right)d\theta \quad .$$

In fact, to $\mathrm{Re}(m/m+1)$ we associate the unique operator

$$\mathrm{Re}(m/m+1)^* : L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi] \to L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$$

satisfying

$$\left\langle \mathrm{Re}(m/m+1)_f, g\right\rangle = \left\langle f, \mathrm{Re}(m/m+1)_g^*\right\rangle,$$

or, in other words,

$$\int_{-\pi}^{\pi} \mathrm{Re}(m/m+1)_f(t)\; g(t)\,dt = \int_{-\pi}^{\pi} f(\theta)\,\mathrm{Re}(m/m+1)_g^*(\theta)\;d\theta$$

for all $f, g \in L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$.It follows, *from Fubini's Theorem*, that

$$\int_{-\pi}^{\pi} \operatorname{Re}(m/m+1)_f(t)\, g(t)\ dt = \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \operatorname{Re} B_m\left(e^{i\theta}, e^{it}\right) d\theta \quad g(t) dt$$

$$= \int_{-\pi}^{\pi} f(\theta) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \operatorname{Re} B_m\left(e^{i\theta}, e^{it}\right) dt \right) d\theta,$$

and consequently

$$\operatorname{Re}(m/m+1)^*_g(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \operatorname{Re} B_m\left(e^{i\theta}, e^{it}\right) dt \qquad \left(g \in L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]\right).$$

Summarizing, we have proved the

Theorem 2.2.4. *If* $m \geq 0$, *then, for any* $u(z) \in L^2_{\mathbb{R}}(C)$ *and any* $f(t) \in L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$, *it holds*

$$\operatorname{Re}(m/m+1)^*_u(z) = \frac{1}{2\pi i} \int_C u(\zeta) \frac{\operatorname{Re} B_m(z,\zeta)}{z}\ d\zeta$$

and

$$\operatorname{Re}(m/m+1)^*_f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \operatorname{Re} B_m\left(e^{it}, e^{i\theta}\right) d\theta\,.$$

The continuity property of Padé-type operators $\operatorname{Re}(m/m+1)$ can be used to prove new convergence results:

Theorem 2.2.5. (a). *If the sequence* $\left\{ u_n \in L^2_{\mathbb{R}}(C) : n = 0,1,2,... \right\}$ *converges to* $u \in L^2_{\mathbb{R}}(C)$ *in the* L^2-*norm, then*

$$\lim_{n\to\infty} \mathrm{Re}(m/m+1)_{u_n}(z) = \mathrm{Re}(m/m+1)_u(z)$$

in the L^2-*norm*.

(b). *If the sequence* $\left\{ f_n \in L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi] : n = 0,1,2,... \right\}$ *converges to* $f \in L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$ *in the* L^2-*norm, then*

$$\lim_{n\to\infty} \mathrm{Re}(m/m+1)_{f_n}(t) = \mathrm{Re}(m/m+1)_f(t)$$

in the L^2-*norm.*

For series of functions, there is a direct consequence of this *Theorem*:

Corollary 2.2.6. (a). *If the series of functions*

$$u(z) = \sum_{n=0}^{\infty} a_n \cdot u_n(z) \ \left(a_n \in \mathbb{R}, u_n \in L^2_{\mathbb{R}}(C)\right)$$

converges in the L^2-*norm, then*

$$\mathrm{Re}(m/m+1)_u(z) = \sum_{n=0}^{\infty} a_n \,\mathrm{Re}(m/m+1)u_n(z)$$

in the L^2-*norm.*

(b). *If the series of functions*

$$f(t) = \sum_{n=0}^{\infty} a_n \cdot f_n(t) \ \left(a_n \in \mathbb{R}, f_n \in L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]\right)$$

converges in the L^2-*norm then*

$$\operatorname{Re}(m/m+1)_f(t)=\sum_{n=0}^{\infty}a_n \operatorname{Re}(m/m+1)_{f_n}(t)$$

in the L^2-*norm.*

Let us now determine conditions under which the integral operator $\operatorname{Re}(m/m+1)$ is compact onto $L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$. Since, for each fixed $t\in[-\pi,\pi]$, the kernel function $\operatorname{Re}B_m\left(e^{i\theta},e^{it}\right)$ is bounded in θ, it follows, from *Tonelli's Theorem*, that

Theorem 2.2.7. *If there is a constant* $c_*<\infty$ *such that*

$$\int_{-\pi}^{\pi}\left|\operatorname{Re}B_m\left(e^{i\theta},e^{it}\right)\right|^2 d\theta\leq(2\pi)^2 c_*$$

for almost all $t\in[-\pi,\pi]$, *then the Padé-type operator*

$$\operatorname{Re}(m/m+1):L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]\to L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$$

is compact. Moreover

$$\left\|\operatorname{Re}(m/m+1)\right\|\leq(2\pi)^{5/2}\cdot c_*$$

and $\operatorname{Re}(m/m+1)^*$ *is also compact.*

It is readily seen that if the Padé-type operator

$$\operatorname{Re}(m/m+1):L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]\to L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$$

is compact, then it is not one-to-one. This follows from the fact that

$$\dim L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]=\infty\ ,$$

and therefore 0 must be an eigenvalue of $\mathrm{Re}(m/m+1)$.

However, it would be interesting to know necessary and sufficient conditions under which, for any $h\in L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$, there is a $f\in L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$ with

$$\mathrm{Re}(m/m+1)_f=h\,.$$

Of course, a general such a condition is given by the inequality

$$\left\|\mathrm{Re}(m/m+1)^*_f\right\|_2\geq c\ \|f\|_2\,.$$

This inequality can also be written in the form

$$\int_{-\pi}^{\pi}|f(t)|^2dt\leq c\int_{-\pi}^{\pi}\left|\int_{-\pi}^{\pi}f(\theta)\,\mathrm{Re}\,B_m\left(e^{it},e^{i\theta}\right)d\theta\right|^2dt$$

for some constant $c>0$ and any $f\in L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$. Obviously, this inequality holds if and only if

$$|f(t)|\leq c\left|\int_{-\pi}^{\pi}f(\theta)\ \mathrm{Re}\,B_m\left(e^{it},e^{i\theta}\right)\,d\theta\right|$$

for almost all $t\in[-\pi,\pi]$, and thus we have proved the following

Theorem 2.2.8. *If there is a constant* $c>0$ *such that*

$$|f(t)|\leq c\left|\int_{-\pi}^{\pi}f(\theta)\,\mathrm{Re}\,B_m\left(e^{it},e^{i\theta}\right)\,d\theta\right|$$

almost everywhere on $[-\pi,\pi]$, *for every* $f \in L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$, *then the range of* $\mathrm{Re}(m/m+1)$ *equals* $L^2_{\mathbb{R},(2\pi-per)}[-\pi,\pi]$.

Let us finally turn to integral representation formulas in the harmonic case. If u is harmonic and real-valued in the unit disk, then, for any $0 \le r < 1$, the restriction

$$u_r(t) = u(r\,e^{it}) \ (-\pi \le t \le \pi)$$

of $u(z)$ to the circle of radius r can be interpreted as a real-valued, $2\pi-$periodic function in $L^2[-\pi,\pi]$. According to *Theorem 2.2.3*, the Padé-type approximant $\mathrm{Re}(m/m+1)_{u_r}(t)$ to $u_r(t)$ is given by the integral representation formula:

$$\begin{aligned}\mathrm{Re}(m/m+1)_{u_r}(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u_r(\theta)\,\mathrm{Re}\left\{4\pi\,\overline{G}_m\left(r\,e^{i\theta}, r\,e^{it}\right)-1\right\}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u_r\left(r\,e^{i\theta}\right)\mathrm{Re}\left\{4\pi\,\overline{G}_m\left(r\,e^{i\theta}, r\,e^{it}\right)-1\right\}d\theta.\end{aligned}$$

After applying the simple change of variables

$$z = r\,e^{it} \text{ and } \zeta = r\,e^{i\theta},$$

we obtain

$$\begin{aligned}\mathrm{Re}(m/m+1)_u(z) &= \frac{1}{2\pi i}\int_{|\zeta|=r} u(\zeta)\,\frac{\mathrm{Re}\left\{4\pi\,\overline{G}_m(\zeta,z)-1\right\}}{\zeta}\,d\zeta \\ &= \frac{1}{2\,\pi i}\int_{|\zeta|=r} u(\zeta)\,\frac{\mathrm{Re}\,B_m(\zeta,z)}{\zeta}\,d\zeta,\end{aligned}$$

and hence we can state the following

Theorem 2.2.9. *Let*

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m\geq 0, 0\leq k\leq m}$$

be an infinite triangular interpolation matrix with complex entries and, for any $m \geq 0$. *Let also* $G_m(x,z)$ *be the unique polynomial of degree at most* m *which interpolates the function* $(1-xz)^{-1}$ *at* $x = \pi_{m,0}, \pi_{m,1}, \pi_{m,2}, \ldots, \pi_{m,m}$ *(*z *is fixed and* $\left|\pi_{m,k}\right| < 1$ *whenever* $k \leq m$*).*

The Padé-type approximant $\mathrm{Re}(m/m+1)_u(z)$ *to the harmonic real-valued function* $u(z)$ *in the disk is given by the following integral representation formula:*

$$\mathrm{Re}(m/m+1)_u(z) = \frac{1}{2\pi i}\int_{|\zeta|=z} \frac{u(\zeta)}{\zeta}\,\mathrm{Re}\,B_m(\zeta,z)\,d\zeta \qquad (z \in D).$$

As it is mentioned in *Paragraph 1.2.1*, the function $\mathrm{Re}(m/m+1)_u(z)$ is the real part of an analytic function in the unit disk, and therefore, it is a harmonic real-valued function in D (of course, under the assumption $\left|\pi_{m,k}\right| < 1$ for all $k \leq m$). If $H_{\mathbb{R}}(D)$ is the space of all harmonic real-valued functions in D, the integral operator

$$\mathrm{Re}(m/m+1) : H_{\mathbb{R}}(D) \to H_{\mathbb{R}}(D)$$

$$: u(z) \mapsto \mathrm{Re}(m/m+1)_u(z) = \frac{1}{2\pi i}\int_{|\zeta|=|z|} \frac{u(\zeta)}{\zeta}\,\mathrm{Re}\,B_m(\zeta,z)\;d\zeta$$

is said to be a *Padé-type operator of* $H_{\mathbb{R}}(D)$.

It is easily seen that a Padé-type operator of $H_{\mathbb{R}}(D)$ is continuous. For, if $\{u_n \in H_{\mathbb{R}}(D) : n = 0,1,2,\ldots\}$ and

$$\lim_{n\to\infty} u_n = u \in H_{\mathbb{R}}(D)$$

compactly in the disk D, then, by the maximum principle for harmonic functions, we have

$$\begin{aligned}\sup_{|z|\le r}&\left|\operatorname{Re}(m/m+1)_{u_n}(z)-\operatorname{Re}(m/m+1)_u(z)\right|\\&=\sup_{|\zeta|=r}\left|\operatorname{Re}(m/m+1)_{u_n}(z)-\operatorname{Re}(m/m+1)_u(z)\right|\\&=\frac{1}{2\pi}\sup_{|z|=r}\left|\int_{|\zeta|=r}[u_n(\zeta)-u(\zeta)]\frac{\operatorname{Re}B_m(\zeta,z)}{\zeta}\,d\zeta\right|\\&\le\frac{1}{2\pi r}\,2\pi r\left\{\sup_{|z|=r,|\zeta|=r}\left|\operatorname{Re}B_m(\zeta,z)\right|\right\}\left\{\sup_{|\zeta|=r}\left|u_n(\zeta)-u(\zeta)\right|\right\}\\&\le L(r,m)\left\{\sup_{|\zeta|=r}\left|u_n(\zeta)-u(\zeta)\right|\right\}\end{aligned}$$

for any $r<1$. Hence, the continuity of $\operatorname{Re}(m/m+1):H_{\mathbb{R}}(D)\to H_{\mathbb{R}}(D)$ follows.

As for the L^2 – case, the continuity of the Padé-type operator for $H_{\mathbb{R}}(D)$ leads to some interesting convergence results.

Theorem 2.2.10. *If the sequence $\{u_n : n=0,1,2,...\}$ of harmonic real-valued functions in the open unit disk converges compactly to $u\in H_{\mathbb{R}}(D)$, then there holds*

$$\lim_{n\to\infty}\operatorname{Re}(m/m+1)_{u_n}(z)=\operatorname{Re}(m/m+1)_u(z)$$

compactly in D.

Corollary 2.2.11. *If the series of harmonic real-valued functions*

$$u(z)=\sum_{n=0}^{\infty}a_n\,u_n(z)\qquad\left(a_n\in\mathbb{R},\,u_n\in H_{\mathbb{R}}(D)\right)$$

converges compactly in the disk, then

$$\operatorname{Re}(m/m+1)_u(z) = \sum_{n=0}^{\infty} a_n \operatorname{Re}(m/m+1)_{u_n}(z)$$

the convergence of the series being compact in D.

Remark 2.2.12. In [23], Brezinski showed that the (Hermite) interpolation polynomial $G_m(x,z)$ of $(1-xz)^{-1}$ at $x = \pi_{m,0}, \pi_{m,1}, \dots, \pi_{m,m}$ is given by

$$G_m(x,z) = \frac{1}{1-xz}\left(1 - \frac{V_{m+1}(x)}{V_{m+1}(z^{-1})}\right) \qquad \left(z \neq \pi_{m,k}^{-1}, k = 0,1,\dots,m\right),$$

where $V_{m+1}(x)$ is any generating polynomial

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m}\left(x - \pi_{m,k}\right) \ (\gamma \neq 0).$$

We thus obtain the following analytic expressions for the two kernels

$$\frac{\operatorname{Re} B_m(\zeta,z)}{\zeta} \text{ and } \operatorname{Re} B_m\left(e^{i\theta}, e^{it}\right):$$

$$\frac{\operatorname{Re} B_m(\zeta,z)}{\zeta} = \zeta^{-1} \operatorname{Re}\left\{\frac{4}{1-\zeta\,\overline{z}}\left(1 - \overline{z}^{m+1}\prod_{k=0}^{m}\frac{\zeta - \overline{\pi_{m,k}}}{1 - \overline{z\,\pi_{m,k}}}\right) - 1\right\}$$

and

$$\operatorname{Re} B_m\left(e^{i\theta}, e^{it}\right) = \operatorname{Re}\left\{\frac{4\pi}{1 - e^{i(\theta - t)}}\left(1 - \prod_{k=0}^{m}\frac{e^{i\theta} - \overline{\pi_{m,k}}}{e^{it} - \overline{\pi_{m,k}}}\right) - 1\right\},$$

respectively.

The investigation of more useful and simple expressions for these two kernels, as well as their deeper properties remain to be studied and could be constitute an interesting direction of research.

2.2.2. Integral Representations and Composed Padé-Type Operators

We are now in position to generalize definitions and results of *Paragraph 2.2.1* to the context of composed Padé-type approximation.

Set

$$L_{\mathbb{C}}^{p}(C) := \{u \in L^{p}(C) : u \text{ is complex} - \text{valued function}\}$$

$$L_{\mathbb{C},(2\pi-per)}^{p}[-\pi,\pi] := \{f \in L^{p}[-\pi,\pi] : f \text{ is complex} - \text{valued and } 2\pi - \text{periodic}(: f(-\pi) = f(\pi)) \text{ function}\}$$

and

$$H_{\mathbb{C}}(D) := \{u : D \to C : u \text{ is harmonic and complex} - \text{valued function}\}.$$

From *Theorems 2.2.3* and *2.2.9*, it follows immediately the

Theorem 2.2.13. *For* $j = 1,2$, *let*

$$\mathrm{M}^{(j)} = \left(\pi_{m,k}^{(j)}\right)_{m\geq 0, 0\leq k\leq m}$$

be an infinite triangular interpolation matrix with complex entries $\pi_{m,k}^{(j)} \in D$, *and, for any* $m \geq 0$, *let* $G_m^{(j)}(x,z)$ *be the unique polynomial of degree at most* m *which interpolates the function* $(1-xz)^{-1}$ *at* $x = \pi_{m,0}^{(j)}, \pi_{m,1}^{(j)}, \ldots, \pi_{m,m}^{(j)}$ (z *is regarded as a parameter*). *If*

$$G_m^{(j)}(x,z) = \sum_{v=0}^{m} g_v^{(j,m)}(z)\, x^v ,$$

we denote by $\overline{G_m^{(j)}}(x,z)$ *the polynomial*

$$\sum_{v=0}^{m} \overline{g_v^{(j,m)}(z)}\, x^v .$$

Put

$$B_m^{(j)}(x,z) = 4\pi\, \overline{G_m^{(j)}}(x,z) - 1 .$$

(a). *For any* $u = u_1 + i \cdot u_2 \in L^2_{\mathbb{C}}(C)$, *the corresponding composed Padé-type approximant* $(m/m+1)_u(z)$ *to* $u(z)$ *has the following integral representation*

$$(m/m+1)_u(z) = \frac{1}{2\pi i} \int_C \left\{ u_1(\zeta) \frac{\operatorname{Re} B_m^{(1)}(\zeta, z)}{\zeta} + i\, u_2(z) \frac{\operatorname{Re} B_m^{(2)}(\zeta, z)}{\zeta} \right\} d\zeta$$

$(|z| = 1)$. *Equivalently*

$$(m/m+1)_u(e^{it}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ u_1(e^{i\theta}) \operatorname{Re} B_m^{(1)}(e^{i\theta}, e^{it}) + i\, u_2(e^{i\theta}) \operatorname{Re} B_m^{(2)}(e^{i\theta}, e^{it}) \right\} d\theta$$

$(-\pi \leq t \leq \pi)$.

(b). *For any* $f = f_1 + i\, f_2 \in L^2_{\mathbb{C},(2\pi-per)}[-\pi, \pi]$, *the corresponding composed Padé-type approximant* $(m/m+1)_f(t)$ *to* $f(t)$ *has the following integral representation*

$$(m/m+1)_f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ f_1(\theta) \operatorname{Re} B_m^{(1)}(e^{i\theta}, e^{it}) + i\, f_2(\theta) \operatorname{Re} B_m^{(2)}(e^{i\theta}, e^{it}) \right\} d\theta$$

$(-\pi \leq t \leq \pi)$.

(c). *For any* $u = u_1 + i u_2 \in H_{\mathbb{C}}(D)$, *the corresponding composed Padé-type approximant* $(m/m+1)_u(z)$ *to* $u(z)$ *has the following integral representation*

$$(m/m+1)_u(z) = \frac{1}{2\pi i} \int_{|\zeta| = |z|} \left\{ u_1(z) \frac{\operatorname{Re} B_m^{(1)}(\zeta, z)}{\zeta} + i u_2(\zeta) \frac{\operatorname{Re} B_m^{(2)}(\zeta, z)}{\zeta} \right\} d\zeta$$

$(|z| < 1)$.

In view of *Theorem 1.2.15*, we can also give integral representation for classical Padé-type approximants to analytic functions:

Corolary 2.2.14. *Let*

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m\geq 0, 0\leq k\leq m}$$

be an infinite triangular interpolation matrix with complex entries $\pi_{m,k} \in D$, *and, for any* $m \geq 0$, *let* $G_m(x,z)$ *be the unique polynomial of degree at most* m *which interpolates the function* $(1-xz)^{-1}$ *at* $x = \pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}$ (z *is regarded as a parameter*).

If

$$G_m(x,z) = \sum_{v=0}^{m} g_v^{(m)}(z)\, x^v ,$$

denote by $\overline{G_m}(x,z)$ *the polynomial*

$$\sum_{v=0}^{m} \overline{g_v^{(m)}(z)}\, x^v ,$$

and put

$$B_m(x,z) := 4\pi\, \overline{G_m}(x,z) - 1 .$$

For any $f \in \mathcal{O}(D)$, *the corresponding Padé-type approximant* $(m/m+1)_f(z)$ *to* $f(z)$ (in the Brezinski sense of [20]) *has the following integral representation*

$$(m/m+1)_f(z) = \frac{1}{2\pi i} \int_{|\zeta|=|z|} f(\zeta)\, \frac{\operatorname{Re} B_m(\zeta,z)}{\zeta}\, d\zeta \qquad (|z|<1).$$

Under the assumptions of *Theorem 2.2.13*, each one of the following integral operators

$$(m/m+1): L^2_{\mathbb{C}}(C) \to L^2_{\mathbb{C}}(C):$$

$$u = u_1 + iu_2 \mapsto (m/m+1)_u(z)$$

$$= \frac{1}{2\pi i}\int_c \left\{ u_1(\zeta)\frac{\operatorname{Re} B_m^{(1)}(\zeta,z)}{\zeta} + i\, u_2(\zeta)\frac{\operatorname{Re} B_m^{(2)}(\zeta,z)}{\zeta}\right\} d\zeta,$$

$$(m/m+1): L^2_{\mathbb{C},(2\pi-per)}[-\pi,\pi] \to L^2_{\mathbb{C},(2\pi-per)}[-\pi,\pi]:$$

$$f = f_1 + i f_2 \mapsto (m/m+1)_f(t)$$

$$= \frac{1}{2\pi}\int_{-\pi} \left\{ f_1(\theta)\operatorname{Re} B_m^{(1)}\left(e^{i\theta},e^{it}\right) + i\, f_2(\theta)\operatorname{Re} B_m^{(2)}\left(e^{i\theta},e^{it}\right)\right\} d\theta$$

and

$$(m/m+1): H_{\mathbb{C}}(D) \to H_{\mathbb{C}}(D):$$

$$u = u_1 + i u_2 \mapsto (m/m+1)_u(z)$$

$$= \frac{1}{2\pi i}\int_{|\zeta|=|z|} \left\{ u_1(\zeta)\frac{\operatorname{Re} B_m^{(1)}(\zeta,z)}{\zeta} + i\, u_2(\zeta)\frac{\operatorname{Re} B_m^{(2)}(\zeta,z)}{\zeta}\right\} d\zeta$$

is called a *composed Padé-type operator for* $L^2_{\mathbb{C}}, L^2_{\mathbb{C},(2\pi-per)}[-\pi,\pi]$ and $H_{\mathbb{C}}(D)$, respectively.

Under the assumptions of *Corollary 2.2.14*, the integral operator

$$(m/m+1): \mathcal{O}(D) \to \mathcal{O}(D): f \mapsto (m/m+1)_f(z) = \frac{1}{2\pi i}\int_{|\zeta|=|z|} f(\zeta)\frac{\operatorname{Re} B_m(\zeta,z)}{\zeta}d\zeta$$

is called a *Padé-type operator for* $\mathcal{O}(D)$.

The continuity property for these integral operators follows directly from arguments cited in *Paragraph 2.2.1* and leads to some new convergence results:

Theorem 2.2.15. *Under the assumptions and notations of Theorem 2.213 and Corollary 2.2.14,*

(a). *if the sequence* $\{u_n \in L^2_{\mathrm{C}}(C) : n = 0,1,2,...\}$ *converges to* $u \in L^2_{\mathrm{C}}(C)$ *in the* $L^2 -$ *norm, then*

$$\lim_{n\to\infty}(m/m+1)_{u_n}(z) = (m/m+1)_u(z)$$

in the $L^2 -$ *norm;*

(b). *if the sequence* $\{f_n \in L^2_{\mathrm{C},(2\pi-per)}[-\pi,\pi] : n = 0,1,2,...\}$ *converges to* $f \in L^2_{\mathrm{C},(2\pi-per)}[-\pi,\pi]$, *with respect to the* $L^2 -$ *norm, then*

$$\lim_{n\to\infty}(m/m+1)_{f_n}(t) = (m/m+1)_f(t)$$

in the $L^2 -$ *norm;*

(c). *if the sequence* $\{u_n \in H_{\mathrm{C}}(D) : n = 0,1,2,...\}$ *converges to* $u \in H_{\mathrm{C}}(D)$ *compactly in D, then*

$$\lim_{n\to\infty}(m/m+1)_{u_n}(z) = (m/m+1)_u(z)$$

compactly in D;

(d). *if the sequence* $\{f_n \in \mathcal{O}(D) : n = 0,1,2,...\}$ *converges to* $f \in \mathcal{O}(D)$ *compactly in D,*

then

$$\lim_{n\to\infty}(m/m+1)_{f_n}(z) = (m/m+1)_f(z)$$

compactly in D.

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Especially, for series of functions, we have the following immediate consequence:

Corollary 2.2.16. *Under the assumptions of Theorem 2.2.13 and Corollary 2.2.14,*

(a). *if the series of functions*

$$u(z)=\sum_{n=0}^{\infty} a_n u_n(z) \ \left(a_n \in \mathbb{C}, u_n \in L^2_{\mathbb{C}}(C)\right)$$

converges in the $L^2 -$*norm, then*

$$(m/m+1)_u(z)=\sum_{n=0}^{\infty} a_n \ (m/m+1)_{u_n}(z)$$

in the $L^2 -$*norm;*

(b). *if the series of functions*

$$f(t)=\sum_{n=0}^{\infty} a_n \ f_n(t) \ \left(a_n \in \mathbb{C}, f_n \in L^2_{\mathbb{C},(2\pi-per)}[-\pi,\pi]\right)$$

converges in the L^2*-norm, then*

$$(m/m+1)_f(t)=\sum_{n=0}^{\infty} a_n \ (m/m+1)_{f_n}(t)$$

in the $L^2 -$*norm;*

(c). *if the series of functions*

$$u(z)=\sum_{n=0}^{\infty} a_n \ u_n(z) \ \left(a_n \in \mathbb{C}, u_n \in H_{\mathbb{C}}(D)\right)$$

converges compactly in the disk D, then

$$(m/m+1)_u(z)=\sum_{n=0}^{\infty} a_n \ (m/m+1)_{u_n}(z)$$

compactly in D;

(d). *if the series of analytic functions*

$$f(z) = \sum_{n=0}^{\infty} a_n \; f_n(z) \;\; (a_n \in \mathbb{C}, \; f_n \in \mathcal{O}(D))$$

converges compactly in D, then

$$(m/m+1)_f(z) = \sum_{n=0}^{\infty} a_n \; (m/m+1)_{f_n}(z)$$

compactly in D.

Remark 2.2.17. Padé and Padé-type approximants to arbitrary series of functions were first considered by Brezinski in [20] and [26].

Chapter 3

Higher Dimensional Analogous: Generalized Padé and Padé-Type Approximation and Integral Representations

Summary

For complex dimensions greater than one, the most highly appreciated theorems on rational approximation have no obvious analogous. Further, the "iterated" Padé and Padé-type theory is based on the multidimensional Cauchy kernel and leads to extremely complicated computations. In this *Chapter*, we will replace the multidimensional Cauchy kernel by the Bergman kernel function $K_\Omega(x,z)$, and we will define generalized Padé and Padé-type approximants to any f in the space $OL^2(\Omega)$ of analytic functions on Ω which are of class L^2. The characteristic property of these approximants is that their Fourier series representations with respect to some orthonormal basis for $OL^2(\Omega)$ match the Fourier series expansion of f as far as possible. After studying the error formulas and the convergence problems related to this approximation, we will show that generalized Padé-type approximants have integral representations that give rise to the consideration of an integral operator, the so-called generalized Padé-type operator. This operator maps every $f \in OL^2(\Omega)$ to a generalized Padé-type approximant to f. By using the continuity property of this operator, we will obtain convergence results about series of analytic functions of class L^2. Next, we will discuss extensions for the notion of generalized Padé-type approximation to continuous functions on a compact set E in R^n satisfying Markov's inequality (M_2) with respect to some measure or classical Markov's inequality (M_∞). Further, we will give integral representations and consider generalized Padé-type operators for the space $C(E)$ of all continuous functions on E. Our study will conclude with an extension of these ideas into every functional Hilbert space H and also with the definition and properties of generalized Padé-type approximants to a linear operator of H into itself. As an application, we will

prove a Painlevé-type theorem in case of arbitrary bounded open sets in C^n, and we will give numerical examples making use of generalized Padé-type approximants.

Introduction

The principal aim of this *Chapter* is to present a generalization of Padé and Padé-type approximation theory in several variables.

Translating a result from one complex variable to several is more involved than merely saying "Now, let $n>1$". Indeed, many arguments in one variable use the Taylor power series expansion of analytic functions into the open disks. In several variables, the open polydisks do not enjoy a very elevated status and the domains of convergence of the power series representations exhibit a much greater variety than in one variable. On the other hand, if $n>1$, the ring $\mathbb{P}(\mathbb{C}^n)$ of complex analytic polynomials in $\mathbb{C}^n$ is not principal and henceforth it is not an Euclidean ring. This means that whenever $n>1$ there is no division process in $\mathbb{P}(\mathbb{C}^n)$, which in particular implies that the cherished notion of continued fraction is absent from the theory of functions of several complex variables. Furthermore, in contrast to the one variable setting, there is no facility in the management of a logical connection between two apparently related mathematical entities: the polynomial of $\mathbb{C}^n$ and its degree.

So, if one would like to adapt the simple proofs in one variable to the case of several variables, then three major obstacles present themselves. First, the local representation of a function analytic into a domain in $\mathbb{C}^n$ by its Taylor series may lead to extremely complicated and difficult computations. Second, the polydisk does not qualify to be the general target domain because of the failure of the property to be the maximal domain of convergence of a multiple power series. Finally, there is no division process in $\mathbb{P}(\mathbb{C}^n)$, when $n>1$.

Since, because of all these reasons, many of the most highly appreciated theorems on rational approximation have no obvious analogue in several complex variables, one might expect that the theory of Padé and Padé-type approximants in $\mathbb{C}^n$ lacks the appeal of the classical one variable theory. We want to show how some of our favorite results in one complex variable can be viewed in order to obtain interesting generalizations in several variables. More precisely, we shall show how the proofs of multidimensional «rational» approximation theory can be cleared of their dependence on the polydisks and the methods of Taylor series and reconnected to Brezinski's original ideas, on Padé-type approximation ([19], [20], [21], [22] and [23]).

It is reasonable to suspect that the outlet lies with the consideration of another type of series representation for analytic functions. By the classical theory of Hilbert spaces, every function which is analytic and of class L^2 into a bounded open domain in $\mathbb{C}^n$ has a Fourier series representation in terms of an orthonormal basis. Thus, the natural perspective that comes to mind is simply this one: we will be able to extend Padé-type approximation theory to analytic functions, which are of class L^2 into a bounded open subset of $\mathbb{C}^n$, if we know a few of its Fourier coefficients. Notice that a great gain in this new approach will be the global validity of our approximation results.

We have probably convinced the reader that the treatment of such a global method is limited to the special class of analytic L^2 functions. We will feel better, if the reader discerns that analogous definitions and results hold also for continuous functions on a compact set in R^n verifying a Markov property, and, more generally, for the elements of any functional Hilbert space H, as well as for any linear operator $H \to H$. This extension insures the powerful global character of the above ideas.

In some sense, the development of these methods will manage to cut us off from the roots of Padé and of Padé-type approximation. We certainly do not advocate abandoning the beautiful machines that have been developed. What we do advocate is a reinvestigation of the basic directions of the subject.

Since most numerical analysts are not familiar with complex analysis in $\mathbb{C}^n$, *Section 3.1*, will first collect some standard definitions, terminology and results of the theory of several complex variables. Next, *Section 3.2* will give a different approach to classical Padé-type approximation in many complex dimensions. For this purpose, we may observe that, when the complex dimension n is equal to 1, the most important kernel in integral representations of analytic functions is the *Cauchy kernel* $(1-xz)^{-1}$; any Padé and Padé-type approximation in one complex variable is based on the free choice of polynomials interpolating this kernel. For complex dimensions n greater than 1, the classical "iterated" Padé and Padé-type theory, developed briefly in *Section 3.1*, is based on interpolations of the *multidimensional Cauchy kernel function* $(1-x_1z_1)^{-1}...(1-x_nz_n)^{-1}$ in polydisks. Unfortunately, this "iterated" approximation process is restrained only into polydisks and leads to extremely complicated computations. So, in *Paragraph 3.2.3* of *Section 3.2*, we will replace the multidimensional Cauchy kernel by *the Bergman kernel function* into an arbitrary open bounded set Ω in $\mathbb{C}^n$:

$$K_\Omega(z,x)=K_\Omega(z_1,...,z_n,x_1,...,x_n).$$

As it will be mentioned in *Paragraph 3.2.1* of this *Section*, the Bergman kernel function $K_\Omega(z,x)$ belongs to the Hilbert space $OL^2(\Omega)$ of all functions that are analytic and of class L^2 in Ω. For any orthonormal basis $\{\varphi_j : j=0,1,2,...\}$ for $OL^2(\Omega)$, one has the representation

$$K_\Omega(z,x)=\sum_{j=0}^{\infty}\varphi_j(z)\overline{\varphi_j(x)},$$

whenever $z\in\Omega$ and $x\in\Omega$. Our idea is then to replace $K_\Omega(z,x)$ by simpler interpolating expressions consisting of generalized polynomials.

To do so, for any $m=0,1,2,...,$ we will consider the $(m+1)-$dimensional complex vector space $\overline{\Phi_{m+1}}$ which is generated by the Tchebycheff system $\{\overline{\varphi_0},\overline{\varphi_1},...,\overline{\varphi_{m+1}}\}$and suppose that $\overline{\Phi_{m+1}}$ satisfies the Haar condition into a finite set of pair-wise distinct points $\mathrm{M}_{m+1}=\{\pi_{m,0},\pi_{m,1},...,\pi_{m,m}\}\subset\Omega$ with

$$\mathrm{M}_{m+1}\cap\bigcup_{0\le j\le m}Ker\overline{\varphi_j}=\varnothing.$$

(In other words, we will suppose that every function in $\overline{\Phi_{m+1}}$ has at most m roots in M_{m+1}. For further information about Haar's condition the reader is refereed to the *Paragraph 3.3.2* of

Section 3.2.) Then, for any fixed point $z \in \Omega$, there is a unique generalized polynomial

$$g_m(x,z) = \sum_{j=0}^{m} c_j^{(m)}(z)\overline{\varphi_j(x)} \in \overline{\Phi_{m+1}},$$

such that

$$g_m(\pi_{m,k}, z) = K_\Omega(z, \pi_{m,k}), \text{ for any } k \le m .$$

The Bergman kernel function $K_\Omega(z,x)$ is then replaced by interpolating generalized polynomials $g_m(x,z)$, and, by using approximate quadrature formulas, we will define generalized Padé-type approximation to any $f \in OL^2(\Omega)$: the function

$$\sum_{j=0}^{m} a_j^{(f)} c_j^{(m)}(z) \in OL^2(\Omega)$$

is said to be a *generalized Padé–type approximant to* f, *with generating system* $\mathrm{M}_{m+1} = \{\pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}\}$. Here $a_j^{(f)}$ is the j^{th} order's Fourier coefficient of f with respect to the basis $\{\varphi_j : j = 0,1,2,\ldots\}$:

$$a_j^{(f)} = \int_\Omega f\, \overline{\varphi_j}\, dV .$$

Further, as it can be easily shown,

$$c_j^{(m)}(z) = \sum_{k=0}^{m} \frac{K_\Omega(z, \pi_{m,k})}{\overline{\varphi_j(\pi_{m,k})}} \quad (0 \le j \le m).$$

The terminology used here is due to H.Van Rossum, who in [141] was first introduced the notion of generalized Padé approximants.

The characteristic property of a generalized Padé-type approximant to $f \in OL^2(\Omega)$ is the following one: if

$$\sum_{j=0}^{\infty} b_j^{(m,f)} \varphi_j(z)$$

is the Fourier series expansion of a generalized Padé-type approximant with respect to the basis

$\{\varphi_j : j = 0,1,2,...\}$, then

$$b_j^{(m,f)} = a_j^{(f)}, \text{ for any } j \leq m.$$

After studying error formulas and convergence problems related to such an approximation, we will show that the generalized Padé-type approximants have integral representations which give rise to the consideration of an integral operator, the so-called generalized Padé-type operator

$$OL^2(\Omega) \to OL^2(\Omega): f \mapsto \int_\Omega f(x) \left[\sum_{k=0}^{m} K_\Omega(z,x) \sum_{j=0}^{m} \frac{\overline{\varphi_j(x)}}{\overline{\varphi_j(\pi_{m,k})}} \right] dV(x).$$

It maps every $f \in OL^2(\Omega)$ to a generalized Padé-type approximant to f. By the continuity property of this operator, we obtain convergence results about series of analytic functions of class L^2.

Next, *Paragraph 3.3.1* of *Section 3.3* will deal with brief presentations of some basic material needed in the sequel. More precisely, we will remind fundamental results about Fourier representations of continuous functions on compact subsets of R^n verifying a Markov inequality, and, we will give an expository reference to the main classes of compact sets having this property. *Paragraph 3.3.2* will contain a natural extension of the notion of generalized Padé-type approximation to continuous functions on a compact set $E \subset \mathrm{R}^n$ satisfying Markov's inequality (M_2) with respect to a measure on E. The characteristic property of such an approximation is exactly the same with the corresponding one for the case of analytic L^2-functions into an open bounded subset of R^n. As in the analytic L^2-setting, each generalized Padé-type approximant to a continuous function (on a compact subset of R^n satisfying (M_2) with respect to a measure) has an integral representation, and thus, one can again define an integral operator, the so-called *generalized Padé-type operator for the space* $C^\infty(E)$ of all continuous function of class C^∞ on E:

$$C^\infty(E) \longrightarrow C^\infty(E).$$

This operator maps every continuous function of class C^{∞} on E to a generalized Padé-type approximant to this function. Application of this operator furnishes useful convergence results. Comparatively with the analytic L^2 setting described in *Paragraph 3.2.3* of *Section 3.2*, the continuous case will impose the *self-summability* property for the orthonormal basis $\{\varphi_j : j = 0,1,2,...\}$ in $L^2(E,\mu)$. Finally, in *Paragraph 3.3.2*, it will be shown how one can approximate in the generalized Padé-type sense every function $u \in C^{\infty}(E)$, whenever the compact set E satisfies Markov's classical inequality (M_{∞}).

In *Section 3.4*, our discussion will proceed with an extension of these approximation methods into every functional Hilbert space H. One purpose of this *Section* will be the suitable representation of generalized Padé-type approximants to elements of H, and, on the other hand, the consequent definition of generalized Padé-type approximation to any linear operator $H \to H$. A second general purpose will be the study of the convergence behavior for a sequence of generalized Padé-type approximants to an element of H or to a linear operator $H \to H$.

The last *Section* is devoted to applications and examples. In *Paragraph 3.5.1*, by using generalized Padé-type approximants to the Bergman projection operator, we will give an extension of *Painlevé's Theorem* in the case of arbitrary bounded open sets in $\mathbb{C}^n$. Finally, in *Paragraph 3.5.2*, two numerical examples will be considered making use of certain generalized Padé-type approximants.

3.1. Preliminaries

3.1.1. Some Well Known Results in Several Complex Variables

This *Paragraph* begins at an elementary level with standard definitions and terminology, followed by a systematic brief discussion of the various fundamental concepts of complex convexity related to the remarkable extension properties of analytic functions in more than one variable. It then continues with a comprehensive introduction to Padé-type approximation in many dimensions, and concludes with complete proofs of substantial local and global convergence results.

The general theory of analytic functions of several complex variables was formulated considerably later than the more familiar theory of analytic functions of a single complex variable. Already by the middle of the 19th century, Georg Griedrich Bernhardt Riemann had recognized that the description of all complex structures on a given compact surface involved complex multidimensional moduli spaces. Before the end of the century, Karl Theodor Wilhelm Weierstrass (1815-1897) and Jules Henri Poincaré (1854-1912) had laid the foundation of the local theory and generalized important global results about analytic functions from regions in the complex plane to product domains in $\mathbb{C}^2$ or in $\mathbb{C}^n$. In 1906, F. Hartogs (1874-1943) discovered domains in $\mathbb{C}^2$ with the property that all functions analytic on it necessarily extend analytically to a strictly larger domain, and it rapidly became clear that an understanding of this new phenomenon, which does not appear in one complex variable, would be a central problem in multidimensional function theory([75],[76]). But in spite of major contributions by F. Hartogs, E.E. Levi ([95],[96]), K. Reinhardt, S. Bergman, H. Behnke, H. Cartan, P. Thullen([32]), A. Weil, and others, the principal global problems were still unsolved by the mid 1930s. We emphasize the work of A. Weil, who generalized in 1935 the Cauchy integral formula to polynomial polyhedra in $\mathbb{C}^n$ and obtained an analogue of the Runge approximation theorem for such polyhedra([145]). The peculiarities of several complex variables were well exposed and the central difficulties clearly stated by the time of the appearance of the book [8] of H. Behnke and P. Thullen, but the main problems were still there. Then, K. Oka, equipped with the Weil formula, introduced some brilliant new ideas, and from 1936 to 1951 he systematically solved all the so-called fundamental problems (Cousin problems, Levi problem,...) one after the other([113]). However, K. Oka's work had much more far-reaching implications. In 1940, H. Cartan began to investigate certain algebraic notions implicit in Oka's work, and in the years thereafter, he and K.Oka, independently, began to widen and deepen the algebraic foundations of the theory, building upon

K. Weierstrass' Preparation Theorem([30]). By the time the ideas of H. Cartan and K.Oka became widely known in the early 1950s, they had been reformulated by H. Cartan and J.P. Serre in the language of sheaves introduced in 1945 by J. Leray. During the 1950s and early 1960s, these new methods and tools were used with great success by H. Cartan, J. P. Serre, H. Grauert, R. Remmert, and many others in building the foundation for the general theory of *complex spaces*, i.e., the appropriate higher dimensional analogues of Riemann surfaces([69];[70]). The phenomenal progress made in those years simply overshadowed the more constructive methods present in K.Oka's work up to 1942, and to the outsider, Several Complex Variables seemed to have become a new abstract theory which had little in common with classical complex analysis. In the sixties, L. Hörmander, J. J. Kohn and C. B. Morrey deduced the main results of K. Oka with the help of methods from the theory of partial differential equations and obtained, in addition, estimates in certain weighted L^2-metrics for solutions of the Cauchy-Riemann equations([82]],[85],[86]). Around 1968-69, G. M. Henkin and E. Ramirez -in his dissertation written under H. Grauert- introduced *Cauchy-type Integral Formulas* on strictly pseudoconvex domains. These formulas, and their application shortly thereafter by Grauert, Lieb and Henkin to solving the *Cauchy-Riemann equations* with supremum norm estimates, set the stage for the solution of hard analysis problems during the 1970s([79],[80]). In the seventies, integral representations turned out to be the natural method for solving several problems related to K. Oka's theory, which are connected with the boundary behavior of analytic functions. The basic tool is an integral representation formula for analytic functions discovered in 1955 by J. J. Leray, which contains the Weil formula as a special case. A complete review on the integral representation formulas for analytic functions is given in [124]. Certain developments of this formula made it possible to solve several of such problems that are not easily obtained with other methods. Moreover, it turned out that by means of these formulas one can build up a large part of the theory of functions of several complex variables in a new and more constructive way. During the 1980´s these developments led to a renewed and rapidly increasing interest in Several Complex Variables by analysts with widely differing backgrounds.

Let us first present some of the basic notation in several complex variables. For $n=1,2,...,$ the $n-$dimensional complex number space $\mathbb{C}^n=\{z=(z_1,z_2,...,z_n): z_j \in \mathbb{C}$ $for\ j=1,2,...,n\}$ is the Cartesian product of n copies of $\mathbb{C}$. The classical Hermitian inner product of $\mathbb{C}^n$ is defined by

$$\langle z,w\rangle=\sum_{j=1}^{n} z_j \overline{w_j} \quad (z,w \in \mathbb{C}^n).$$

The associated norm $|z|=\langle z,z\rangle^{1/2}$ induces the Euclidean Metric. The *open ball* of radius $\rho>0$ and center $z \in \mathbb{C}^n$ is defined by $B^n(z,\rho)=\{w\in\mathbb{C}^n: |z-w|<\rho\}$. The collection of balls $\{B^n(z,\rho): \rho\ \textit{is a rational positive number}\}$ forms a countable neighborhood basis at z for the topology of $\mathbb{C}^n$. The topology of $\mathbb{C}^n$ is thus identical with the one arising from the identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. In fact, given any $z=(z_1,z_2,...,z_n)\in\mathbb{C}^n$, each complex coordinate z_j can be written as $z_j=x_j+iy_{i\,j}$ with $x_i, y_i \in \mathbb{R}$ (i is always the imaginary unit $\sqrt{-1}$); the mapping $\mathbb{C}^n \longrightarrow \mathbb{R}^{2n}: z \mapsto (x_1,y_1,x_2,y_2,...,x_n,y_n)$ establishes an $\mathbb{R}$-linear isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, which is compatible with the metric structures : a ball $B^n(z,\rho)$ in $\mathbb{C}^n$ is identified with an Euclidean ball in $\mathbb{R}^{2n}$ of equal radius ρ, and conversely. Often it is convenient to use another system of neighborhoods: the *open polydisk* (or *open polycylinder*) $\Delta^n(z,r)$ of multiradius $r=(r_1,r_2,...,r_n), r_j>0$, and center $z\in\mathbb{C}^n$ is the Cartesian product of n open disks in $\mathbb{C}$:

$$\Delta^n(z,r)=\{w=(w_1,w_2,...,w_n)\in\mathbb{C}^n: |z_j-w_j|<r_j, for\ j=1,2,...,n\}$$

$$=\Delta^1(z_1,r_1)\times\Delta^2(z_2,r_2)\times ... \times\Delta^n(z_n,r_n).$$

More generally, a *polydomain* is the Cartesian product of n planar domains. A set $\Omega\subset\mathbb{C}^n$=$\mathbb{R}^{2n}$ is *open*, if for every $z\in\Omega$ there is a ball $B^n(z,\rho)\subset\Omega$ or a polydisk $\Delta^n(z,r)\subset\Omega$.

For an open set $\Omega \subset \mathbb{C}^n$ and $d \in \mathbb{N} \bigcup \{\infty\}$, $C^d(\Omega)$ denotes the space of d times continuously differentiable complex valued functions on Ω; we also write $C(\Omega)$ instead of $C^0(\Omega)$. For a $f \in C^d(\Omega)$, with $d < \infty$, we define the C^d-*norm* of f over Ω by

$$\|f\|_{C^d(\Omega)} = \sum_{|\nu| \le d \, (\nu \in \mathrm{N}^{2n})} \sup_{z \in \Omega} \left| D^{\nu} f(z) \right|,$$

where D^{ν} is the differential operator

$$D^{\nu} = \frac{\vartheta^{|\nu|}}{\vartheta x_1^{\nu_1} \vartheta y_1^{\nu_2} \ldots \vartheta x_n^{\nu_{2n-1}} \vartheta y_n^{\nu_{2n}}} \qquad \left(|\nu| = \nu_1 + \nu_2 + \ldots + \nu_{2n} \right).$$

The space $\left\{ f \in C^d(\Omega) : \|f\|_{C^d(\Omega)} < \infty \right\}$ is complete in C^d-norm $\|\cdot\|_{C^d(\Omega)}$, and hence it is a Banach space. Similarly, if Ω is bounded, the space

$$C^d\left(\overline{\Omega}\right) = \left\{ f \in C^d(\Omega) : D^{\nu} f \textit{ extends continuously to } \overline{\Omega} \textit{ for all } \nu \in \mathrm{N}^{2n} \textit{ with } |\nu| = \sum_{j=1}^{2n} \nu_j \le d \right\}$$

with norm

$$\|f\|_{C^d(\overline{\Omega})} = \sum_{|\nu| \le d \, (\nu \in \mathrm{N}^{2n})} \sup_{z \in \overline{\Omega}} \left| D^{\nu} f(z) \right|, \qquad \left(f \in C^d\left(\overline{\Omega}\right) \right)$$

is also a Banach space. A function $f : \Omega \to \mathbb{C}$ is said to be *analytic* (or *holomorphic*) on Ω, if $f \in C^1(\Omega)$ and f satisfies the following system of partial differential equations

$$(hCR) \qquad \frac{\vartheta f}{\vartheta \overline{z}_j}(z) = 0 \quad \text{for } 1 \le j \le n \text{ and } z \in \Omega,$$

where we have used the notation

$$\frac{\vartheta}{\vartheta \overline{z}_j} = \frac{1}{2} \cdot \left(\frac{\vartheta}{\vartheta x_j} - \frac{1}{i} \frac{\vartheta}{\vartheta y_j} \right).$$

The most elementary examples of analytic functions on $\mathbb{C}^n$ are the analytic polynomials

$$p(x) = p(x_1, x_2, ..., x_n) = \sum_{\nu_1=0}^{k_1} ... \sum_{\nu_n=0}^{k_n} a^{(p)}_{\nu_1,...,\nu_n} \ x_1^{\nu_1} ... x_n^{\nu_n} ,$$

with $a^{(p)}_{\nu_1,...,\nu_n} \in \mathbb{C}$. The multi-index $k = (k_1, ..., k_n) \in \mathbb{N}^n$ is the degree of $p(x)$. We will denote by $\mathbb{P}(\mathbb{C}^n)$ the space of all analytic polynomials in $\mathbb{C}^n$. Equation (hCR) is called *the system of homogeneous Cauchy-Riemann equations*.

Let us give another representation for the solutions of the homogeneous Cauchy-Riemann equations. Let Ω be an open subset of $\mathbb{C}^n$ and let $f \in C^1(\Omega)$; its differential $(df)_z$ at $z \in \Omega$ is the unique $\mathbb{R}$-linear map $\mathbb{R}^{2n} \to \mathbb{R}^2$ which approximates f near z in the sense that

$$f(\zeta) = f(z) + (df)_z(\zeta - z) + \mathrm{o}(|\zeta - z|).$$

In terms of the real coordinates $(x_1, y_1, ..., x_n, y_n)$ of $\mathbb{C}^n = \mathbb{R}^{2n}$, one has

$$(df)_z = \sum_{j=1}^{n} \frac{\vartheta f}{\vartheta x_j}(z)(dx_j)_z + \sum_{j=1}^{n} \frac{\vartheta f}{\vartheta y_j}(dy_j)_z$$

where $d x_j$ and $d y_j$ are the differentials of the real coordinate functions. Via the identifications $\mathbb{R}^{2n} = \mathbb{C}^n$ and $\mathbb{R}^2 = \mathbb{C}$, the differential $(df)_z$ can be viewed as a map $\mathbb{C}^n \to \mathbb{C}$ which is $\mathbb{R}$-linear, though not necessarily $\mathbb{C}$-linear. In particular, the differentials $d x_j$ and $d y_j$ are not linear over $\mathbb{C}$. One therefore considers the differentials $d z_j = d x_j + i d y_j$ (this is $\mathbb{C}$-linear) and $d \overline{z_j} = d x_j - i d x_j$ (this is conjugate $\mathbb{C}$-linear) of the complex coordinate functions. A simple computation shows that

$$(df)_z = \sum_{j=1}^{n} \frac{\vartheta f}{\vartheta z_j}(z)(dz_j)_z + \sum_{j=1}^{n} \frac{\vartheta f}{\vartheta \overline{z_j}}(d\overline{z_j})_z$$

were we have introduced the operator

$$\frac{\vartheta}{\vartheta z_j} = \frac{1}{2}.\left(\frac{\vartheta}{\vartheta x_j} + \frac{1}{i}\frac{\vartheta}{\vartheta y_j} \right), \text{ for } 1 \le j \le n .$$

It is directly verified that a function $f \in C^1(\Omega)$ is analytic on Ω if

$$(df)_z = \sum_{j=1}^{n} \frac{\vartheta f}{\vartheta z_j}(z)(dz_j)_z$$

or equivalently, if and only if its differential $(df)_z$ at z is a $\mathbb{C}$-linear map, whenever $z \in \Omega$. Furthermore, it is obvious that any function $f \in C^1(\Omega)$ fulfilling (hCR) satisfies the *Cauchy-Riemann equations* in the z_j-coordinate for any j, and hence is analytic in each variable separately. It is a remarkable phenomenon of complex analysis that, conversely, any function $f : \Omega \to \mathbb{C}$ which is analytic in each variable separately is analytic in Ω ([71]). This shows that the requirement $f \in C^1(\Omega)$ can be dropped in the definition of analyticity.

The set of all functions analytic on Ω will be denoted by $O(\Omega)$. It is closed under point-wise addition and multiplication. We will always consider $O(\Omega)$ equipped with the natural topology in which convergent sequences are precisely those which converge compactly on Ω. As in the case of one complex variable, the basic local properties of analytic functions follow from the *Cauchy Integral Formula* on polydisks : if $f \in C\left(\overline{\Delta^n(z,r)}\right) \cap O\left(\Delta^n(z,r)\right)$, then

$$(CIF) \quad f(w) = (2\pi i)^{-n} \int_{|\zeta_j - z_j| = r_j \ (j=1,2,\ldots,n)} \frac{f(\zeta)}{(\zeta_1 - w_1)^{\nu_1+1} \ldots (\zeta_n - w_n)^{\nu_n+1}} d\zeta_1 \ldots d\zeta_n ,$$

for $w = (w_1, \ldots, w_n) \in \Delta^n(z,r)$.

Applying *Cauchy's Integral Formula* to $\Delta^n(z,\delta) \subset\subset \Delta^n(z,r)$ and differentiating under the sign integral, we obtain

$$D^\nu f(z) = \nu! \, (2\pi i)^{-n} \int_{|\zeta_j - z_j| = r_j \ (j=1,2,\ldots,n)} \frac{f(\zeta)}{(\zeta_1 - w_1)^{\nu_1+1} \ldots (\zeta_n - w_n)^{\nu_n+1}} d\zeta_1 \ldots d\zeta_n ,$$

(with $\nu! = \nu_1!...\nu_n!$). One can, of course, extend the *Cauchy Integral Formula* for analytic functions with continuous extension on the distinguished boundary $b\Omega = \vartheta\Omega_1 \times ... \times \vartheta\Omega_n$ of a polydomain $\Omega = \Omega_1 \times ... \times \Omega_n$. It is also readily seen that for any $\nu \in \mathbb{N}^n$, $1 \le p \le \infty$ and $U \subset\subset \Omega$, there is a constant $c = c(\nu, p, U, \Omega)$ such that

$$\sup_{z \in U} \left| D^\nu f(z) \right| \le c \left\| f \right\|_{L^p(\Omega)}, \text{ for all } f \in O(\Omega) \cap L^p(\Omega).$$

The space $O(\Omega) \cap L^p(\Omega)$ of analytic L^p −functions on Ω will be denoted by $OL^p(\Omega)$. The *Cauchy Integral Formula* (CIF) implies also strong convergence results: the classical *Weierstrass* and *Montel Theorems* have natural generalizations in several variables. Moreover, one can show that, via *Cauchy's Integral Formula*, every analytic function can be represented locally by a convergent power series.

To see this, we first remind the basic facts about multiple series, that is, formal expressions

$$\sum_{\nu \in \mathrm{N}^\nu} \beta_\nu = \sum_{\nu_1,...,\nu_n=0}^{\infty} \beta_{\nu_1,...,\nu_\nu} \quad (\beta_\nu = \beta_{\nu_1,...,\nu_n} \in \mathbb{C}).$$

The multiple series $\sum_{\nu \in \mathrm{N}^\nu} \beta_\nu$ is called *convergent* if

$$\sum_{\nu \in \mathrm{N}^n} \left| \beta_\nu \right| = \sup\left\{ \sum_{\nu \in \Lambda} \left| \beta_\nu \right| : \Lambda \textit{ is finite} \right\} < \infty .$$

It is well known that the convergence of

$$\sum_{\nu \in \mathrm{N}^n} \left| \beta_\nu \right|,$$

as defined above, is necessary and sufficient for the following to hold: given any bijection $\sigma : \mathbb{N} \to \mathbb{N}^n$, the ordinary series

$$\sum_{j=0}^{\infty} \beta_{\sigma(j)}$$

converges in the usual sense to a limit $B \in \mathbb{C}$ which is independent of σ; this number B is called the *sum* (or *limit*) *of the multiple series*, and one writes

$$B = \sum_{\nu \in N^n} \beta_\nu .$$

A power series in n complex variables $\zeta_1, \zeta_2, ..., \zeta_n$ centered at the point $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$ is a multiple series

$$\sum_{\nu \in N^n} \beta_\nu$$

with general term

$$\beta_\nu = c_\nu (\zeta - z)^\nu = c_{\nu_1, \nu_2, ..., \nu_n} (\zeta_1 - z_1)^{\nu_1} (\zeta_2 - z_2)^{\nu_2} ... (\zeta_n - z_n)^{\nu_n} ,$$

where $c_\nu \in \mathbb{C}^n$ for $\nu \in \mathbb{N}^n$. Without loss of generality, we will only consider multiple power series centered at $z = 0$. The domain of convergence of the power series

$$\sum_{\nu \in N^n} c_\nu \zeta^\nu = \sum_{\nu_1, ..., \nu_n = 0}^{\infty} c_{\nu_1, ..., \nu_n} \zeta_1^{\nu_1} ... \zeta_n^{\nu_n}$$

is the interior of the set of all points $\zeta \in \mathbb{C}^n$ for which this series converges. Many familiar results from the theory of ordinary series have easy extensions to multiple series. For example, the following result generalizes *Abel's Lemma* and gives the basic general behavior of convergent power series: suppose $c_\nu \in \mathbb{C}$ for $\nu \in \mathbb{N}^n$ and that for some $w \in \mathbb{C}^n$ there holds

$$\sup_{\nu \in N^n} |c_\nu w^\nu| < \infty ;$$

letting $r = (|w_1|, |w_2|, ..., |w_n|)$, the power series

$$\sum_{\nu \in N^n} c_\nu \zeta^\nu$$

converges on the polydisk $\Delta^n(0, r)$. From *Weierstrass' Theorem*, it follows that a power series

$$f(\zeta) = \sum_{\nu \in N^n} c_\nu \zeta^\nu ,$$

with nonempty domain of convergence Ω defines an analytic function $f \in O(\Omega)$. The domain of convergence Ω of the power series

$$\sum_{\nu \in \mathbb{N}^n} c_\nu \zeta^\nu$$

is a (possibly empty) complete Reinhardt domain. Recall that a *Reinhardt domain* Ω *with center* 0 is an open circled (around 0) set in $\mathbb{C}^n$, in the sense that for every $\zeta \in \Omega$ the torus

$\{w \in \mathbb{C}^n : w = (\zeta_1 e^{i\theta_1}, ..., \zeta_n e^{i\theta_n}), \textit{ with } 0 \le \theta_j \le 2\pi \textit{ whenever } j = 1, ..., n\}$

lies in Ω as well; a Reinhardt domain Ω is complete if for every $\zeta \in \Omega$ one has

$$\Delta^n(0, (|\zeta_1|, ... |\zeta_n|)) \subseteq \Omega.$$

We now show that, for every $f \in O(\Delta^n(z,r))$, the Taylor series expansion of f at z converges to f on $\Delta^n(z,r)$:

$$(Tse) \qquad f(w) = \sum_{\nu \in \mathbb{N}^n} \frac{D^\nu f}{\nu!}(z)(w-z)^\nu, \text{ for } w \in \Delta^n(z,r).$$

In the *Cauchy Integral Formula* (CIF), applied to $w \in \Delta^n(z,\delta) \subset\subset \Delta^n(z,r)$, one expands $(\zeta - w)^{-1} = (\zeta_1 - w_1)^{-1} ... (\zeta_n - w_n)^{-1}$ into a multiple geometric series

$$(Gs) \qquad (\zeta - w)^{-1} = \sum_{\nu \in \mathbb{N}^n} \frac{(w-z)^\nu}{(\zeta - z)^{\nu+1}}.$$

This series converges uniformly for $\zeta \in b\Delta^n(z,\delta)$, since

$$(|w_j - z_j| / |\zeta_j - z_j|) \le (|w_j - z_j| / \delta_j) < 1$$

for such any ζ and all $1 \le j \le n$. It is therefore legitimate to substitute (Gs) into (CIF) and to interchange summation and integration, leading to (Tse):

$$f(w)=\sum_{\nu\in\mathbb{N}^n}\left[(2\pi i)^{-n}\int_{\substack{|\zeta_j-z_j|=\delta_j\\(j=1,2,\dots,n)}}\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta_1\dots d\zeta_n\right](w-z)^{\nu}=\sum_{\nu\in\mathbb{N}^n}\frac{D^{\nu}f(z)}{\nu!}(w-z)^{\nu}.$$

The following results are easy generalizations of the corresponding classical one variable results:

* if there is a nonempty open set $U\subset\Omega$, such that $f(z)=0$ for all $z\in U$, and if Ω is connected, then $f\equiv 0$ on Ω (:*Identity Theorem*);
* if $|f|$ has a local maximum at the point $z\in\Omega$, then f is constant on Ω (:*Maximum Principle*).

Next, let us consider a map $F:\Omega\to\mathbb{C}^m$ (Ω is always an open set in $\mathbb{C}^n$). By writing $F=(f_1,f_2,\dots,f_m)$ and $f_k=u_k+iv_k$, where u_k and v_k are real valued functions on Ω, we can view $F=(u_1,v_1,u_2,v_2,\dots,u_m,v_m)$ as a map from $\Omega\subset\mathbb{R}^{2n}$ into $\mathbb{R}^{2m}$. If F is differentiable at $z\in\Omega$, its differential $(df)_z$: $\mathbb{R}^{2n}\to\mathbb{R}^{2m}$ is a $\mathbb{R}$-linear transformation with matrix representation given by the real Jacobian matrix

$$J_{\mathbb{R}}(F)=\begin{pmatrix}\frac{\vartheta u_1}{\vartheta x_1}\ \frac{\vartheta u_1}{\vartheta y_1}\cdots\frac{\vartheta u_1}{\vartheta x_n}\ \frac{\vartheta u_1}{\vartheta y_n}\\ \dots\dots\dots\dots\dots\dots\\ \frac{\vartheta v_m}{\vartheta x_1}\ \frac{\vartheta v_m}{\vartheta y_1}\cdots\frac{\vartheta v_m}{\vartheta x_n}\ \frac{\vartheta v_m}{\vartheta y_n}\end{pmatrix}\quad (:2m\times 2n \text{ matrix})$$

evaluated at z. The map $F=(f_1,f_2,\dots,f_m):\Omega\to\mathbb{C}^m$ is called *analytic*, if its complex coordinates $f_1,f_2,\dots,f_m$ are analytic functions on Ω. The composition of two analytic maps is again an analytic map (:*chain rule*). If F is analytic, its differential $(dF)_z$ at z is a $\mathbb{C}$-linear transformation with complex matrix representation

$$J_{\mathbb{C}}(F) = \begin{pmatrix} \dfrac{\vartheta f_1}{\vartheta z_1} \dfrac{\vartheta f_1}{\vartheta z_2} \cdots \dfrac{\vartheta f_1}{\vartheta z_{n-1}} \dfrac{\vartheta f_1}{\vartheta z_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\vartheta f_m}{\vartheta z_1} \dfrac{\vartheta f_m}{\vartheta z_2} \cdots \dfrac{\vartheta f_m}{\vartheta z_{n-1}} \dfrac{\vartheta f_m}{\vartheta z_n} \end{pmatrix} \quad (: m \times n \text{ matrix})$$

evaluated at z; it is easily verified that the analyticity of F implies that

$$\det J_{\mathbb{R}}(F) = |\det J_{\mathbb{C}}(F)|^2.$$

If, moreover, $F:\Omega \to \mathbb{C}^m$ is *nonsingular* at $z \in \Omega$ (,i.e., $\det J_{\mathbb{C}}(F)(\zeta) \neq 0$), then there are open neighborhoods U_z of z and $U_{F(z)}$ of $F(z)$, such that

$$[F/U_z]: U_z \to U_{F(z)}$$

is a homeomorphism with analytic inverse

$$[F/U_z]^{-1} : U_{F(z)} \to U_z .$$

Motivating by this result, we may give the following *Definition*: given two open sets $\Omega \subseteq \mathbb{C}^n$ and $\Omega' \subseteq \mathbb{C}^m$, we say that the map $F:\Omega \to \Omega'$ is *bianalytic* if F is an analytic homeomorphism with analytic inverse $F^{-1}:\Omega' \to \Omega$. If F is bianalytic, it follows from the chain rule that $[J_{\mathbb{C}}(F^{-1})](F(z))$ is the inverse matrix of $[J_{\mathbb{C}}(F)](z)$; in particular, F is nonsingular on Ω (,i.e., $\det[J_{\mathbb{C}}(F)](z) \neq 0$ at every $z \in \Omega$) and $m = n$. If $F = (f_1, f_2,..., f_n):\Omega \to \Omega'$ is bianalytic, we also say that $(f_1, f_2,..., f_n)$ is an *analytic* (or *complex*) *coordinate system* on Ω . We are now in position to introduce a local generalization of the concept of complex linear subspace of $\mathbb{C}^n$ which is invariant under complex coordinate changes : a set $M \subseteq \mathbb{C}^n$ is called a *complex submanifold* of $\mathbb{C}^n$, if for every point $z \in M$ there are an analytic coordinate system $(f_1, f_2,..., f_n)$ on a neighborhood U_z at z, and an integer k ($0 \leq k \leq n$), such that $M \cap U = \{z \in U_z : f_j(z) = 0 \ for \ j > k\}$; the integer k is called the *complex dimension* of M

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at z and is denoted by $k=\dim_{\mathbb{C}} M_z$. Notice that $\dim_{\mathbb{C}} M_z$ is independent of the analytic coordinate system $(f_1, f_2, ..., f_n)$ and that $\dim_{\mathbb{C}} M_z$ is locally constant on M and hence is constant on each connected component of M. The *dimension* of M is defined by $\dim_{\mathbb{C}} M=\sup_{z\in M} \dim_{\mathbb{C}} M_z$. Every open set $\Omega \subset \mathbb{C}^n$ is a complex submanifold of $\mathbb{C}^n$. For another example of complex submanifold, one can consider the zero set $Z(f,\Omega)=\{z\in\Omega : f(z)=0\}$ of an analytic function f defined on the open set Ω: if Ω is connected and if $Z(f,\Omega)\neq\varnothing$, $f\neq 0$, then there exists an open set $D\subset\Omega$ such that $Z(f,D)$ is a nonempty complex submanifold of D of dimension $n-1$. In contrast to the one complex dimension case, the zeroes of a function analytic in 2 or more variables are never isolated. Let us give a useful description of complex submanifolds: a subset M of $\mathbb{C}^n$ is a complex submanifold if and only if for every $z\in M$ there are a neighborhood U_z of z, an open ball $B^k(a,\varepsilon)\subset\mathbb{C}^k$, and a nonsingular analytic map. $H: B^k(a,\varepsilon)\to\mathbb{C}^n$ such that $H(B^k(a,\varepsilon))=M\cap U_z$. The map H is called a *local parametrization* of M at z. A function $f: M\to\mathbb{C}$ is called *analytic at* $z\in M$ if $f\circ H^{-1}$ is analytic into an open neighborhood of $H^{-1}(z)$, (for a local parametrization H of M at z); f is said to be *analytic on* M if it is analytic at every $z\in M$. The reader familiar with differentiable submanifolds of $\mathbb{R}^n$ (, i.e., curves, surfaces, e. t. c.) will have recognized the obvious formal similarities between those concepts and the theory of complex submanifolds. But, there are surprising differences as well, as evidenced by the following result, which has no counterpart for differentiable or even analytic submanifolds of $\mathbb{R}^n$: any compact complex submanifold of $\mathbb{C}^n$ consists of finitely many points. More surprising is the fact that an injective analytic map F from an open subset Ω of $\mathbb{C}^n$ into $\mathbb{C}^n$ is necessarily nonsingular, and hence bianalytic from Ω onto $F(\Omega)$. The proof of this fact is a consequence of some technical elementary information about the zero set of analytic functions. No comparable result exists in real calculus.

In several complex variables it is important to study not just the zero set of one analytic function, but also of several analytic functions, i.e., of analytic maps. As it is pointed out, the case of nonsingular analytic maps leads us to the concept of a complex submanifold. The general case is quite a bit more complicated. A subset A of the region $\Omega \subset \mathbb{C}^n$ is called *analytic* in Ω if A is closed in Ω and if for every $p \in \mathrm{A}$ there are an open neighborhood U_p of p in Ω and an analytic map $H_p : U_p \to \mathbb{C}^{m_p}$ such that $U_p \cap \mathrm{A} = \{z \in U_p : H_p(z) = 0\}$. A point $p \in \mathrm{A}$ of an analytic set A is called a *regular point* of A if there is a neighborhood W_p of p, such that $\mathrm{A} \cap W_p$ is a complex submanifold of W_p, and a s*ingular point* otherwise. The set of regular points of A is denoted by $R(\mathrm{A})$: it is the maximal complex submanifold contained in A. Every (closed in Ω) submanifold M of $\Omega \subset \mathbb{C}^n$ is analytic in Ω, with $R(\mathrm{A}) = \mathrm{A}$; in particular, Ω itself is analytic in Ω. Each analytic subset A of a connected region $\Omega \subset \mathbb{C}^n$ is thin, (and hence $\Omega - \mathrm{A}$ is connected) or equal to Ω. For further studies, the interested reader should consult some of the specialized literature, for example [70], [71], or [107].

Let us now present a brief discussion of a phenomenon, which distinguishes, more than anything else, function theory in several variables from the classical one-variable theory. In 1906, Hartogs discovered a serious and fundamental difference between the case of one and many complex dimensions stating that if K is a compact subset of an open set Ω in $\mathbb{C}^n$, $n \geq 2$, such that $\Omega - K$ is connected, then every function f analytic in $\Omega - K$ can be extended analytically into Ω ([75]). The methodology of Hartogs's *Proof* (which was adopted also by Bochner in 1943, when he referred to the general problem of analytic and meromorphic extension ([12])) is based on the use of Cauchy's integral formula or Stokes' theorem. In 1961, Ehrenpreis claimed that this phenomenon is attributed to the behavior of the functions-solutions of the inhomogeneous Cauchy-Riemann equations ([54]). Recall that the *inhomogeneous system of Cauchy-Riemann equations* is the following:

$$(inhCR) \qquad \frac{\vartheta F}{\vartheta \overline{z_j}} = u_j \ , \ \text{for } 1 \le j \le n,$$

where $u_1, u_2, ..., u_n$ are given C^1 – functions defined on an open subset Ω of $\mathbb{C}^n$. For $n = 1$, this system is degenerated to an equation with one unknown complex function f, or equivalently to a system of 2 real equations with unknowns the real functions $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$. On the contrary, when $n \ge 2$, the system $(inhCR)$ is over-determinate, because in this case we have more equations than unknowns. It is easily seen that if there exists a solution $f \in C^2(\Omega)$ for $(inhCR)$ then the functions $u_1, u_2, ..., u_n$ must necessarily satisfy the following integral conditions:

$$(\Sigma) \qquad \frac{\vartheta u_\kappa}{\vartheta \overline{z}_j} = \frac{\vartheta u_j}{\vartheta \overline{z}_\kappa}, \quad 1 \le j \le n.$$

It is obvious for $n \ge 2$ these restrictive conditions may cause great differences comparatively with the case when $n = 1$. Thus, the problem of solving the inhomogeneous Cauchy-Riemann equations $(inhCR)$ can be formulated equivalently in the following form:

$(\overline{\vartheta} - equation)$

given an open set Ω *in* $\mathbb{C}^n$ *and*

$$u = \sum_{j=1}^{n} u_j \, d\overline{z_j}$$

in $C^1_{(0,1)}(\Omega)$={*differential forms on* Ω, *of type* $(0,1)$ *and with coefficients* $u_j \in C^1(\Omega)$ }, *find* $f \in C^2(\Omega)$ *such that*

$$\overline{\vartheta} f = u,$$

where $\overline{\vartheta}$ is the differential operator

$$\overline{\vartheta} = \sum_{j=1}^{n} \frac{\vartheta}{\vartheta \overline{z_j}} d\overline{z_j}.$$

Notice that, with this terminology, the integral conditions (Σ) are equivalent to the equation $\overline{\vartheta} u = 0$. Historically, the first who presented results in this direction was Oka in 1937 ([113]). Later, in 1965, Hörmander ([82]) used weight functions in order to modify L^2-norms (as Carleman had done for questions of partial differential equations) and presented a complete collection of important facts, achieving to include in his theory some results that have already appeared through the works of Garabedian, Spencer, Morrey, Ask, Kohn and Nirenberg. One of the fundamental theorems of modern Complex Analysis, as is developed based on the ground of partial differential equations, is that *the equation* $\overline{\vartheta} f = u$ *has a solution* $f \in C^{\infty}_{(0,q)}(\Omega)$ *for every* $u \in C^{\infty}_{(0,q+1)}(\Omega)$ *such that* $\overline{\vartheta} u = 0$ *if and only if* Ω *is a domain of holomorphy*. Remind that an open subset Ω of $\mathbb{C}^n$ is called a *domain of holomorphy* if there is no part of the boundary of Ω across which every function analytic in Ω can be continued analytically. Thus, the $\overline{\vartheta}-$equation is solved exclusively (at least for the C^{∞} case) into the open subsets of $\mathbb{C}^n$ which are the biggest domains of definition of analytic functions. In the complex plane every open set is a domain of holomorphy. To see this, it is sufficient to see that for every point p in the boundary $\vartheta\,\Omega$ of an open planar set Ω, the analytic function $(z-p)^{-1}$ can not be extended analytically past the point *p*. We can also prove that

$dist(\hat{K}_{O(\Omega)}, \vartheta\,\Omega) = dist(K, \vartheta\,\Omega)$,

for any $K \subset\subset \Omega$; $dist(A,B)$ is the notation used for $\inf\{dist(a,b) : a \in A, b \in B\}$; $\hat{K}_{O(\Omega)}$ is the *analytic hull* of K defined by $\hat{K}_{O(\Omega)} := \{z \in \Omega : |f(z)| \le \sup_K |f| \ \ for\ all\ \ f \in O\,(\Omega)\}$. So, the analytical (and not only topological) generalization in $\mathbb{C}^n$ of an open planar set is not simply an open set, but a domain of holomorphy in $\mathbb{C}^n$. The attempts and contributions for the characterization of domains of holomorphy began from the first decade of the 20th century. The basic examples of domains which are not domains of holomorphy are $H-$shaped *Hartogs figures* in $\mathbb{C}^2$. These are of the form

$$H_{\varepsilon} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1 + \varepsilon\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : 1 - \varepsilon < |z_1| < 1 + \varepsilon \ and \ |z_2| < 1\}.$$

Any complex function analytic on H_{ε} extends to the convex hull of H_{ε} using *Cauchy's Integral Formula* (CIR). Hartogs analyzed this phenomenon. He considered a slight generalization of this basic Hartogs figure, namely the *Hartogs domains*

$$H_{\varphi} = \{(z_1, z_2) \in \mathbb{C}^2 : \ z_1 \in U : open \ set \ in \ \mathbb{C} \ and \ |z_2| < e^{-\varphi(z_1)}\}.$$

He showed that analytic functions on these generalized H's extend beyond the boundary precisely when the function φ fails to be a *Hartogs function* ([76]). Loosely speaking Hartogs functions are the lattice generated by the functions $c \log|f|$, where c>0 and f is analytic. A Hartogs function is the same as a *subharmonic function* ([77]) –but this was about 15 years before subharmonic functions were invented– and, anyway, this equivalence was not proved until 1956 by Bremermann. This brought the topic of subharmonic functions into the theory of several complex variables. In other words, Hartogs' discovery showed that Hartogs domains in $\mathbb{C}^2$ are domains of holomorphy precisely when φ is a subharmonic (:Hartogs) function. Notice that φ can be interpreted as $-\log(dist_2)$, on the base $U = U * 0$, where $dist_2$ denotes the distance from a point to the boundary in the z_2 $-$direction. It took the genius of Oka to see how this could be used for general domains. The Hartogs domains are, after all, rather special. Oka proved in 1942 that a general domain in $\mathbb{C}^2$ is a domain of holomorphy if and only if the function $-\log(dist)$ is subharmonic on each complex line. Oka called such functions *pseudoconvex* ([113]). This class of functions was simultaneously introduced by Lelong, who called them *plurisubharmonic* ([93]). So to be more precise, an upper semicontinuous function whose restriction to any complex line is subharmonic is said to be *plurisubharmonic*. The terminology of Lelong has prevailed, so these functions are nowdays called plurisubharmonic. Lelong's motivation to introduce the plurisubharmonic functions is rooted in a similar way in the Hartogs extension phenomenon. Levi, in 1910, observed that *a smoothly bounded strongly convex domain* *in* $\mathbb{C}^2$ *is a domain of holomorphy*([95]). Such a domain has a defining function which is strongly convex. The property of being a domain of holomorphy is invariant under bianalytic maps. The bianalytic image of a strongly convex domain is, therefore, a domain of holomorphy. Carrying over the convex defining function gives us a plurisubharmonic defining function. Lelong's motivation for introducing plurisubharmonic functions was that they could be used to describe in this way a complex analytic version of convexity : a region $\Omega \subset \mathbb{C}^n$ is called *plurisubharmonic convex*, if for every compact set $K \subset \Omega$ its *plurisubharmonic convex hull* $\hat{K}_{PS(\Omega)} := \{z \in \Omega : u(z) \leq \sup_K u \text{ for any plurisubharmonic function } u \text{ in } \Omega\}$ is relatively compact in Ω. For a plurisubharmonic convex region Ω in $\mathbb{C}^n$, there holds.

$$\hat{K}_{PS(\Omega)} = \hat{K}_{O(\Omega)}, \text{ whenever } K \subset\subset \Omega.$$

(Here, $\hat{K}_{O(\Omega)}$ denotes, as above, the *holomorphic* (or *analytic*) *hull* of K, that is $\hat{K}_{O(\Omega)} := \{z \in \Omega : |f(z)| \leq \sup_K |f| \text{ for all } f \in O(\Omega)\}$) The statement that a region Ω in $\mathbb{C}^n$ is a domain of holomorphy involves a surrounding space $\mathbb{C}^n$, and thus, it is not clear whether it describes an intrinsic property of Ω. It was a major achievement when in 1932, H. Cartan and Thullen ([29]) discovered an intrinsic characterization of domains of holomorphy in terms of convexity conditions with respect to the algebra of analytic functions. A region $\Omega \subset \mathbb{C}^n$ is called *holomorphically convex*, if for every compact set $K \subset\subset \Omega$ its holomorphic hull $\hat{K}_{O(\Omega)}$ is relatively compact in Ω. According to H. Cartan–Thyllen's theory, *a region* $\Omega \subset \mathbb{C}^n$ *is domain of holomorphy if and only if* Ω *is holomorphically convex*. It follows that *a plurisubharmonic convex region in* $\mathbb{C}^n$ *is a domain of holomorphy*. To prove the converse we must first observe that if there is a plurisubharmonic exhaustion function for the region Ω in $\mathbb{C}^n$, then Ω is plurisubharmonic convex.. Remind that an open set $\Omega \subset \mathbb{C}^n$ is said to be *pseudoconvex* if there is a continuous exhaustion plurisubharmonic function defined on Ω. With this terminology, we can reformulate the above observation: *any pseudoconvex open set in* $\mathbb{C}^n$ *is plurisubharmonic convex*. The identity of domains of holomorphy with pseudoconvex open sets was proved independently by Oka (1953), Norguet (1954) and Bremermann (1954) ([113], [110], [14]). We conclude that *a domain of holomorphy in* $\mathbb{C}^n$ *is a plurisubharmonic convex region*. Summarizing, *the following properties are equivalent for an open subset of* $\mathbb{C}^n$:

(i). Ω *is a domain of holomorphy*,

(ii). Ω *is pseudoconvex*,

(iii). Ω *is plurisubharmonic convex*,

(iv). Ω *is holomorphically convex*,

(v). $-\log(dist(\cdot,\cdot))$ *is plurisubharmonic on* Ω .

In order to present a more complete discussion of pseudoconvexity, we introduce a version of the classical "continuity principle" describing a geometrical very intuitive analogue of linear convexity. If $\Delta \subset\subset \mathbb{C}$ is an open disk and $u : \overline{\Delta} \to \Omega$ is a continuous map which is analytic on Δ, we shall say that $u\left(\overline{\Delta}\right)$ is an *analytic disk S* in Ω and call the set $u(\vartheta\Delta)$ the boundary ϑS of *S*. A region Ω in $\mathbb{C}^n$ is said to satisfy the *continuity principle* if for every family $\{S_a : a \in I\}$ of analytic disks in Ω with

$$\bigcup_{a \in I} \vartheta S_a \subset\subset \Omega ,$$

it follows that

$$\bigcup_{a \in I} S_a \subset\subset \Omega .$$

Regarding

(vi). *For every disk S in* Ω *one has dist* (S, $\vartheta\,\Omega$)=*dist* (ϑS, $\vartheta\,\Omega$),

it is obvious that (vi) implies that

(vii). Ω *satisfies the continuity principle*.

Further, (vii) $\Rightarrow$ (v). Conversely, the maximum principle for subharmonic functions shows that (v) implies (vi). Thus, *properties* (vi) *and* (vii) *give other characterizations for domains of holomorphy*. Property (vi) exhibits once more the strong analogy between pseudoconvexity and linear convexity : just observe that $\Omega \subset \mathrm{R}^{2n}$ is convex if and only if for every line segment $\mathrm{L} \subset \Omega$ one has

$$dist(\mathrm{L}, \vartheta\,\Omega)=dist(\vartheta\,\mathrm{L}, \vartheta\,\Omega).$$

It is natural to ask at this point if there is a «maximal» region $E(\Omega)$ to which every $f \in O(\Omega)$ extends analytically. For example, the bidisk $\Delta^2(0,1)$ clearly is such a maximal region for a Hartogs figure H_ε. The situation is analogous to the problem of finding a «maximal» domain of definition for a single function like $\sqrt{z}$ ($z \in \mathbb{C}$-$\{0\}$), which can only be handled adequately by introducing more abstract spaces, i.e., Riemann surfaces. Similarly, in several variables, one is led to consider domains which have different layers spread over $\mathbb{C}^n$:these are called *Riemann domains*. One can then show that for every domain $\Omega \subset \mathbb{C}^n$, there is a Riemann domain $E(\Omega)$, called the *envelop of holomorphy* of Ω, so that every $f \in O(\Omega)$ has an analytic extension to $E(\Omega)$ and $E(\Omega)$ is «maximal» with respect to this property ([68],[108]). All these remarks suggest that in order to deal with certain global questions it is necessary to extend function theory from domains in $\mathbb{C}^n$ to more abstract spaces. In a pioneering paper [136], published in 1951, K. Stein discovered a class of abstract complex manifolds that enjoy complex analytic properties similar to those of domains of holomorphy. The fundamental importance of these manifolds for global complex analysis was soon recognized, and already in 1952 H. Cartan referred to them as *Stein manifolds* ([30]). Among the axioms which define a Stein manifold X is the requirement that X be holomorphically convex. The other axioms are more technical and they are trivially satisfied for any open set $\Omega \subset \mathbb{C}^n$ or even for any (not necessarily closed) complex submanifold of $\mathbb{C}^n$.

Before continuing our discussion of the various fundamental concepts of «complex convexity», let me mention an approximation theorem ([124]) extending Cartan–Thullen's results. As we have seen, a domain Ω in $\mathbb{C}^n$ is a domain of holomorphy if and only if it is holomorphically convex, that is if and only if $\hat{\mathrm{K}}_{O(\Omega)} \subset\subset \Omega$ for every $\mathrm{K} \subset\subset \Omega$. An open set $\Omega \subset \mathbb{C}^n$ is called a *Runge domain* if the algebra of complex analytic polynomials $\mathbb{P}(\mathbb{C}^n)$ is dense in $O(\Omega)$, when endowed with the Fréchet topology of compact convergence in Ω. More generally, two open sets $\Omega \subset \Omega'$ are called a *Runge pair* if $O(\Omega')$ is dense in $O(\Omega)$. Obviously, Ω is Runge if and only if $(\Omega, \mathbb{C}^n)$ is a Runge pair; moreover, it is classical that *an open set* $\Omega \subset \mathbb{C}$ *is a Runge domain if and only if* Ω *is simply connected. The following statements are equivalent for two domains of holomorphy* $\Omega \subset \Omega'$ *in* $\mathbb{C}^n$ and generalize Cartan–Thullen's fundamental result on holomorphic convexity:

(a). *$O(\Omega')$ is dense in $O(\Omega)$, i.e., (Ω, Ω') is a Runge pair*,

(b). *Ω is $O(\Omega')$-convex, i.e., $\hat{K}_{O(\Omega')} \cap \Omega \subset\subset \Omega$ for every compact set* $K \subset\subset \Omega$.

By the time the ideas of Cartan and Oka became widely known in the early 1950s, they had been reformulated by Cartan and Serre in the language of sheaves. During the 1950s and early 1960s, these new methods and tools were used with great success by Cartan, Serre, Grauert and Remmert and many others in building the foundation for the general theory of *complex spaces*, i.e., the appropriate higher dimensional analogues of Riemann surfaces ([30], [58], [68], [69], [70], [72]). In the late 1960s, the fusion of functional analysis and several complex variables created a new branch of Mathematics: *infinite dimensional analyticity*. The systematic approach for analytic continuation of analytic mappings in infinitely many variables and the original sources of the major ideas and results in this direction are included into the excellent book of G. Coeuré [35] (see also [2], [15], [16], [17], [34], [51], [111], [123], [148] and the bibliography given therein).

We end our brief summary of standard basic properties for analytic functions in open sub-domains in $\mathbb{C}^n$ by defining the very important class of strictly pseudoconvex sets. As it is mentioned above, plurisubharmonic functions are a useful tool in multidimensional complex analysis. The profound conception and orientation of their properties is been determined from the analogy between different kinds of real convexity (for domains and functions) and corresponding concepts in complex case. For a domain $\Omega \subset \mathbb{R}^n$ with C^2 $-$boundary, convexity is characterized by a differential condition in terms of a defining function *r*, as follows:

$(Conv_R 1)$ *if* Ω *is convex near* $p \in \vartheta\,\Omega$, *then the real Hessian* $L_p^R(r,\xi)$ *of r at p is positive semi-definite on the real tangent space* $T_p^R(\vartheta\,\Omega)$ *to* $\vartheta\,\Omega$ *at p*, that is

$$L_p^R(r,\xi) = \sum_{j,k=1}^{n} \frac{\vartheta^2 r}{\vartheta x_j\, \vartheta x_k}(p)\,\xi_j\,\xi_k \geq 0,$$

for all $\xi = (\xi_1,\dots,\xi_n) \in T_p^R(\vartheta\,\Omega) = \left\{\xi \in \mathbb{R}^n : d\,r_p(\xi) = \sum_{j=1}^{n} \frac{\vartheta r}{\vartheta x_j}(p)\xi_j = 0\right\}$.

and

$(Conv_R 2)$ *if* $p \in \vartheta\,\Omega$ *and the real Hessian* $L_p^R(r,\xi)$ *of r at p is strictly positive semi-definite on* $T_p^R(\vartheta\,\Omega)$, that is if

$$L_p^R(r,\xi) = \sum_{j,k=1}^{n} \frac{\vartheta^2 r}{\vartheta x_j\, \vartheta x_k}(p)\,\xi_j\,\xi_k \geq 0,$$

for all $\xi = (\xi_1,\dots,\xi_n) \in T_p^R(\vartheta\,\Omega)$, *then* Ω *is strictly convex at* p (,i.e., there is a convex neighbourhood U_p of p such that $\Omega \cap U_p$ is convex).

In 1910, Levi discovered that domains of holomorphy with C^2 $-$boundary satisfy a complex analogue of $(Conv_R 1)$. More presisely, a domain Ω in $\mathbb{C}^n$ with C^2 $-$boundary is said to be *Levi pseudoconvex at* $p \in \vartheta\,\Omega$, if the *complex Hessian* $L_p^C(r,\xi)$ of the defining function r at p is positive semi-definite on the *complex tangent space* $T_p^C(\vartheta\,\Omega) = T_p^R(\vartheta\,\Omega) \cap i\,T_p^R(\vartheta\,\Omega)$ to $\vartheta\,\Omega$ at p, that is if

$$(LConv_C) \qquad L_p^C(r,\xi) := \sum_{j,k=1}^{n} \frac{\vartheta^2 r}{\vartheta z_j\, \vartheta \overline{z_k}}(p)\,\xi_j\,\overline{\xi_k} \geq 0,$$

for all $\xi = (\xi_1,\xi_2,\dots,\xi_n) \in T_p^C(\vartheta\,\Omega)$. Ω is said to be *Levi pseudoconvex* if the Levi condition

$(LConv_C)$ holds at all points $p \in \vartheta\,\Omega$. Ω is said to be *strictly Levi pseudoconvex at* $p \in \vartheta\,\Omega$ if

$$(SLConv_C) \qquad L_p^C(r,\xi) = \sum_{j,k=1}^{n} \frac{\vartheta^2 r}{\vartheta z_j\, \vartheta \overline{z_k}}(p)\,\xi_j\,\overline{\xi_k} > 0\,,$$

for all $\xi = (\xi_1, \xi_2, ..., \xi_n) \in T_p^C(\vartheta\,\Omega) - \{0\}$; if strict Levi condition $(SLConv_C)$ holds at all points $p \in \vartheta\,\Omega$, then Ω is called *strictly Levi pseudoconvex*. The fundamental Levi's results were:

$(Conv_C 1)$ *if* Ω *is pseudovonvex in* $\mathbb{C}^n$ *with* C^2 *– boundary, then* Ω *is Levi pseudoconvex* ([89], [95]).

and

$(Conv_C 2)$ *if* Ω *is strictly Levi pseudoconvex, then every* $p \in \vartheta\,\Omega$ *has a neighborhood* U_p *such that* $\Omega \cap U_p$ *is pseudoconvex* ([96]).

The first natural question which may be now asked is whether in case of differentiable C^2 – boundaries, Levi pseudoconvexity coincides with pseudoconvexity. The answer is affirmative, so

(viii). (for an open subset Ω of $\mathbb{C}^n$ with C^2 – boundary) Ω *is pseudoconvex if and only if* Ω *is Levi pseudoconvex*.

A second problem that also arises is the description and generalization of strict Levi pseudoconvexity, given that strict Levi pseudoconvexity can be viewed as the exact complex translation of real strict convexity. (To justify this last assertion, we intimate that, for an open set $\Omega \subset \mathbb{C}^n$ with C^2 – boundary near $p \in \vartheta\,\Omega$, Ω is strictly Levi pseudoconvex at p if and only if there is an analytic coordinate system $w = w(z)$ in a neighborhood of p, so that Ω is strictly convex with respect to the w – coordinates; in other words, strict Levi pseudoconvexity is precisely the locally bianalytically invariant formulation of strict Euclidean convexity.) In order to face this second problem, it is important to observe that a function $\phi \in C^2(\Omega)$, defined on an open subset Ω of $\mathbb{C}^n$ (not necessarily with C^2-boundary) is plurisubharmonic if and only if $L_z^C(\phi,\xi)\geq 0$, for all $z\in\Omega$ and all $\xi\in\mathbb{C}^n$. We shall say that the function $\phi\in C^2(\Omega)$ is *strictly plurisubharmonic* if $L_z^C(\phi,\xi)>0$, for all $z\in\Omega$ and all $\xi\in\mathbb{C}^n-\{0\}$. A bounded domain Ω in $\mathbb{C}^n$ is called *strictly pseudoconvex*, if there are a neighborhood $U_{\vartheta\Omega}$ of $\vartheta\ \Omega$ and a strictly plurisubharmonic function $r\in C^2(U_{\vartheta\Omega})$ such that $\Omega\cap U_{\vartheta\Omega}=\{z\in U_{\vartheta\Omega}: r(z)<0\}$. Notice that every bounded planar domain is strictly pseudoconvex and that we do not require that $(dr)_z\neq 0$ for $z\in\vartheta\ \Omega$, so that a strictly pseudoconvex domain does not necessarily have a C^2-boundary. It is easy to see that a strictly pseudoconvex domain is pseudoconvex, and that every bounded subdomain of $\mathbb{C}^n$ with C^2-boundary is strictly Levi pseudoconvex if and only if it is strictly pseudoconvex. It should also be mentionned that most of the well known important developments is complex analysis concern the strictly pseudoconvex case : integral representation formulas ([80], [124]), Fefferman's mapping theorem ([59]), the Chern-Moser invariants ([32]), the work of Henkin and Skoda on zero sets on Nevanlinna functions ([79], [128]), sharp estimates for the $\overline{\vartheta}-$problem, e. t. c.

In order to limit the size of this *Paragraph*, many important topics – for which fortunately excellent references are available–had to be omitted. If, after reading the above condensed, elementary and introductory discussion, you want to learn more about several variables, we highly recommend the text books [13], [60], [69], [72], [82], [87] and [108].

After this brief exposition of some of the standard definitions and results on several complex variables, we are ready to give a first approach to Padé-type approximation in many dimensions. In what follows, our purpose is to define and study the convergence of Padé-type approximants to a function which is analytic into an open polydisk centered at 0. Our discussion will follow the development adopted in [38], [39] and [40].

3.1.2. Classical Padé and Padé-Type Approximants to Analytic Functions of Several Complex Variables

Let

$$\mathrm{M}_1 = \left(\pi^{(1)}_{m_1,k_1}\right)_{m_1\geq 0, 0\leq k_1\leq m_1}, \dots, \mathrm{M}_n = \left(\pi^{(n)}_{m_n,k_n}\right)_{m_n\geq 0, 0\leq k_n\leq m_n}$$

be n infinite triangular matrices with complex entries. For any fixed $z = (z_1,\dots,z_n) \in \mathbb{C}^n$, with

$$z_1 \neq \left(\pi^{(1)}_{m_1,k_1}\right)^{-1} \quad (k_1 = 0,1,\dots,m_1), \quad . \quad . \quad . \quad , z_n \neq \left(\pi^{(n)}_{m_n,k_n}\right)^{-1} \quad (k_n = 0,1,2,\dots,m_n),$$

let

$$Q_{(m_1,\dots,m_n)}(x,z) = Q_{(m_1,\dots,m_n)}(x_1,\dots,x_n,z_1,\dots,z_n)$$

be an analytic polynomial of degree at most $(m_1,\dots,m_n)$ which interpolates $(1-x_1z_1)^{-1}\dots(1-x_nz_n)^{-1}$ in the $(m_1+1)\dots(m_n+1)$ points

$$\left(\pi^{(1)}_{m_1,k_1},\dots,\pi^{(n)}_{m_n,k_n},z_1,\dots,z_n\right),$$

i.e. ,

$$Q_{(m_1,\dots,m_n)}\left(\pi^{(1)}_{m_1,k_1},\dots,\pi^{(n)}_{m_n,k_n},z_1,\dots,z_n\right) = \left(1-\pi^{(1)}_{m_1,k_1}z_1\right)^{-1}\dots\left(1-\pi^{(n)}_{m_n,k_n}z_n\right)^{-1},$$

for any $k_1 = 0,1,\dots,m_1, \dots, k_n = 0,1,\dots,m_n$.

Let Ω be an open subset of $\mathbb{C}^n$, containing 0. Let also $\Delta^n(0,r)$ be the maximal open polydisk of $\mathbb{C}^n$ which is contained in Ω and centered at 0. Let also f be a function of $O(\Omega)$ with Taylor power series expansion around the origin

$$f(z_1,\dots,z_n) = \sum_{\nu_1,\dots,\nu_n=0}^{\infty} a^{(f)}_{\nu_1,\dots,\nu_n} z_1^{\nu_1}\dots z_n^{\nu_n} \quad \left((z_1,\dots,z_n)\in\Delta^n(0,r)\right).$$

It is clear that to f there corresponds a $\mathbb{C}$-linear functional distribution T_f on $\mathbb{P}(\mathbb{C}^n)$ determined by

$$T_f\left(x_1^{\nu_1}\ldots x_n^{\nu_n}\right)=a_{\nu_1,\ldots,\nu_n}^{(f)} \quad \left(\nu_1=0,1,2,\ldots,\ .\ .\ .\ ,\nu_n=0,1,2,\ldots\right).$$

The following *Theorem* is a direct consequence of *Cauchy's Integral Formula* (CIF). The *Proof* is exactly similar to that of [37] or [38] except for the fact that here we need to consider the Banach space

$$A\,(\overline{\Delta^n(0,(\rho_1^{-1},\ldots,\rho_n^{-1}))}):=C\,(\overline{\Delta^n(0,(\rho_1^{-1},\ldots,\rho_n^{-1}))})\cap O\;(\Delta^n(0,(\rho_1^{-1},\ldots,\rho_n^{-1})))$$

instead of the space $O\;(\overline{\Delta^n(0,(\rho_1^{-1},\ldots,\rho_n^{-1}))}):=\{\,f$ *is analytic function in the neighborhood of* $\overline{\Delta^n(0,(\rho_1^{-1},\ldots,\rho_n^{-1}))}\,\}$. This consideration is an essential simplification and its consequences will be appearing in the sequel.

Theorem 3.1.1. *The distribution* T_f *is continuous. Further, for any* $\rho_j<r_j$ $(j=1,2,\ldots,n)$, *there is a continuous extension of* T_f *into* $A\,(\overline{\Delta^n(0,(\rho_1^{-1},\ldots,\rho_n^{-1}))})$.

Proof. Let $\rho=(\rho_1,\ldots,\rho_n)$, $0<\rho_j<r_j$ $(j=1,2,\ldots,n)$. If

$$p(x_1,\ldots,x_n)=\sum_{\nu_1=0}^{k_1}\cdots\sum_{\nu_n=0}^{k_n}\beta_{\nu_1,\ldots,\nu_n}x_1^{\nu_1}\ldots x_n^{\nu_n}$$

is an analytic polynomial in $x=(x_1,\ldots,x_n)$ of degree at most $(k_1,\ldots,k_n)$, then by *Cauchy's Integral Formula* there holds

$$\begin{aligned}\left|T_f(p(x))\right| &= \left|T_f\left(\sum_{\nu_1=0}^{k_1}\cdots\sum_{\nu_n=0}^{k_n}\beta_{\nu_1,\dots,\nu_n}x_1^{\nu_1}\dots x_n^{\nu_n}\right)\right| \\ &= \left|\sum_{\nu_1=0}^{k_1}\cdots\sum_{\nu_n=0}^{k_n}\beta_{\nu_1,\dots,\nu_n}a_{\nu_1,\dots,\nu_n}^{(f)}\right| \\ &= \left|\sum_{\nu_1=0}^{k_1}\cdots\sum_{\nu_n=0}^{k_n}\beta_{\nu_1,\dots,\nu_n}(2\pi i)^{-n}\int_{|\zeta_j|=\rho_j\ (j=1,\dots,n)}\frac{f(\zeta_1,\dots,\zeta_n)}{\zeta_1^{\nu_1+1}\dots\zeta_n^{\nu_n}}d\zeta_1\dots d\zeta_n\right| \\ &\le L(\rho)\sup_{|x_1|\le\rho_1,\dots,|x_n|\le\rho_n}\left|f(x_1,\dots,x_n)\right|\sup_{|x_1|\le\rho_1^{-1},\dots,|x_n|\le\rho_n^{-1}}\left|p(x_1,\dots,x_n)\right|,\end{aligned}$$

where $L(\rho)$ is a constant depending only on ρ. The *Hahn-Banach Theorem* completes now the *Proof.*

The most useful consequence of this *Theorem* is described in the following

Corollary 3.1.2. *For every* $z=(z_1,\dots,z_n)\in\Delta^n(0,r)$, *the number*

$$T_f\left((1-x_1z_1)^{-1}\dots(1-x_nz_n)^{-1}\right)$$

is well defined and equals $f(z)$.

Proof. If $z=(z_1,\dots,z_n)\in\Delta^n(0,r)$, then there is an open polydisk $\Delta^n(0,\rho)=\Delta^n(0,(\rho_1,\dots,\rho_n))\subset\subset\Delta^n(0,r)$ such that $z\in\Delta^n(0,\rho)$. By *Theorem 3.1.1*, the number

$$T_f\left((1-x_1z_1)^{-1}\dots(1-x_nz_n)^{-1}\right)$$

is well defined (: T_f acts on the variable $(x_1,\dots,x_n)\in\overline{(\Delta^n(0;\rho_1^{-1},\dots,\rho_n^{-1}))}$ and z is taken as a parameter). The continuity of the $\mathbb{C}$-linear form T_f implies now that

$$\begin{aligned} f(z) &= \sum_{\nu_1,\ldots,\nu_n=0}^{\infty} a^{(f)}_{\nu_1,\ldots,\nu_n} z_1^{\nu_1} \ldots z_n^{\nu_n} \\ &= \sum_{\nu_1,\ldots,\nu_n=0}^{\infty} T_f\left(x_1^{\nu_1} \ldots x_n^{\nu_n}\right) z_1^{\nu_1} \ldots z_n^{\nu_n} \\ &= T_f\left(\sum_{\nu_1,\ldots,\nu_n=0}^{\infty} x_1^{\nu_1} z_1^{\nu_1} \ldots x_n^{\nu_n} z_n^{\nu_n}\right) \\ &= T_f\left((1-x_1 z_1)^{-1} \ldots (1-x_n z_n)^{-1}\right). \end{aligned}$$

Definition 3.1.3. *The function*

$$T_f\left(Q_{(m_1,\ldots,m_n)}(x,\cdot)\right): \mathbb{C}^n - \left\{(s_1,\ldots,s_n) : s_j = \left(\pi^{(j)}_{m_j,k_j}\right)^{-1} \text{ for } k_j \le m_j \text{ and } j = 1,2,\ldots,n\right\} \to \mathbb{C}:$$

$$z \mapsto T_f\left(Q_{(m_1,\ldots,m_n)}(x,z)\right)$$

is called a Padé-type approximant to f. The analytic polynomial

$$V_{(m_1+1,\ldots,m_n+1)}(x_1,\ldots,x_n) = \gamma \prod_{j=1}^{n} V_{m_j+1}(x_j),$$

with

$$V_{m_j+1}(x_j) = \prod_{k_j=0}^{m_j} \left(x_j - \pi^{(j)}_{m_j,k_j}\right)$$

is the generating polynomial of this approximation ($\gamma \in \mathbb{C} - \{0\}$).

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Before entering into more results, let us show how to construct rational approximations to f. It is well known that the interpolation polynomial of the function $(1-x_1z_1)^{-1}...(1-x_nz_n)^{-1}$ at the points $\left(\pi^{(1)}_{m_1,k_1},...,\pi^{(n)}_{m_n,k_n},z_1,...,z_n\right)$ is given by

$$Q_{(m_1,...,m_n)}(x_1,...,x_n,z_1,...,z_n)=(1-x_1z_1)^{-1}...(1-x_nz_n)^{-1}$$
$$\left(1-\frac{V_{m_1+1}(x_1)}{V_{m_1+1}(z_1^{-1})}\right)...\left(1-\frac{V_{m_n+1}(x_n)}{V_{m_n+1}(z_n^{-1})}\right).$$

Setting

$$\hat{W}_{(m_1,...,m_n)}(z_1,...,z_n):=z_1^{m_1}...z_n^{m_n}\,T_f\left(\left[(-1)^{n+1}V_{m_1+1}(x_1)...V_{m_n+1}(x_n)+\cdots\right.\right.$$
$$\cdots+\sum_{j=1}^{n}V_{m_1+1}(z_1^{-1})...V_{m_{j-1}+1}(z_{j-1}^{-1})V_{m_j+1}(x_j)V_{m_{j+1}+1}(z_{j+1}^{-1})...V_{m_n+1}(z_n^{-1})$$
$$\left.-V_{m_1+1}(z_1^{-1})...V_{m_n+1}(z_n^{-1})\right]$$
$$\left.\times\left[(x_1-z_1^{-1})^{-1}...(x_n-z_n^{-1})^{-1}\right]\right),$$

and

$$\hat{V}_{(m_1+1,...,m_n+1)}(z_1^{-1},...,z_n^{-1}):=z_1^{m_1+1}...z_n^{m_n+1}\,V_{(m_1+1,...,m_n+1)}(z_1^{-1},...,z_n^{-1}),$$

it is easily verified that

$$T_f\left(Q_{(m_1,...,m_n)}(x_1,...,x_n,z_1,...,z_n)\right)$$
$$=\hat{W}_{(m_1,...,m_n)}(z_1,...,z_n)\ /\ \hat{V}_{(m_1+1,...,m_n+1)}(z_1^{-1},...,z_n^{-1}).$$

Thus,

$$T_f\left(Q_{(m_1,...,m_n)}(x_1,...,x_n,z_1,...,z_n)\right)$$

is a rational function in $(z_1,...,z_n)$ of type $((m_1,...,m_n),(m_1+1,...m_n+1))$, which means that it has a numerator with degree in $(z_1,...,z_n)$ at most $(m_1,...,m_n)$ and a denominator with degree in $(z_1,...,z_n)$ at most . $(m_1+1,...,m_n+1)$ Hence, we will make use of the notation

$$(m_1,...m_n / m_1+1,...,m_n+1)_f(z_1,...,z_n) := T_f\left(Q_{(m_1,...,m_n)}(x_1,...,x_n,z_1,...,z_n)\right).$$

A sequence $\{V_{(m_1+1,...,m_n+1)}(x_1,...,x_n) : m_1 = 0,1,2,...,\cdots,m_n = 0,1,2,...\}$ of generating polynomials being given, we now want to study the convergence of the corresponding Padé-type approximation sequence

$$\{(m_1,...,m_n / m_1+1,...,m_n+1)_f(z) : m_j \in \mathrm{N}, \textit{for } j = 1,2,...,n\}.$$

It is readily seen that

$$(m_1,...,m_n / m_1+1,...,m_n+1)_f(z)$$

$$= \sum_{k_1=0}^{m_1} \sigma_{m_1,k_1}^{(1)}(z) \sum_{\nu_1=0}^{k_1}\left(\cdots\left(\sum_{k_n=0}^{m_n} \sigma_{m_n,k_n}^{(n)}(z) \sum_{\nu_n=0}^{k_n} a_{\nu_1,...,\nu_n}^{(f)} z_1^{\nu_1}...z_n^{\nu_n}\right)\cdots\right),$$

where $\sigma_{m_j,k_j}^{(j)}(z)$ $(j = 1,2,...,n)$ is a rational function in $z = (z_1,...,z_n)$ determined by $V_{(m_1+1,...,m_n+1)}(x_1,...,x_n)$. Hence, our problem is equivalent to the following one: «*If*

$$\mathrm{N}_j(z) = \left(\sigma_{m_j,k_j}^{(j)}(z)\right)_{m_j \geq 0, 0 \leq k_j \leq m_j} \qquad (j = 1,2,...,n)$$

are n infinite triangular matrices of complex functions, find the largest open subset of Ω, *into which the* $(\mathrm{N}_1,...,\mathrm{N}_n)-$*transform of the sequence of partial sums of any* $f \in O(\Omega)$ *around the origin*

$$\left\{ \sum_{k_1=0}^{m_1} \sigma_{m_1,k_1}^{(1)}(z) \sum_{\nu_1=0}^{k_1}\left(\cdots\left(\sum_{k_n=0}^{m_n} \sigma_{m_n,k_n}^{(n)}(z) \sum_{\nu_n=0}^{k_n} a_{\nu_1,...,\nu_n}^{(f)} z_1^{\nu_1}...z_n^{\nu_n}\right)\cdots\right) : z = (z_1,...,z_n) \in \mathbb{C}^n,\right.$$

$$\left. m_j \in \mathbb{N} \ (1 \leq j \leq n)\right\}$$

converges compactly to f .»

Let

$\mathbf{D}_{\mathrm{N}_{\lambda_1},...,\mathrm{N}_{\lambda_\mu}}(\Omega) := \{U \subseteq \Omega$:*for any* $f \in O(\Omega)$, *the* $(\mathrm{N}_1,...,\mathrm{N}_n)-$*transform of the* *sequence of partial sums of* f *around* 0, *converges to* f, *compactly on* U, *if* $m_{\lambda_1} \to \infty, \ldots, m_{\lambda_\mu} \to \infty\}$,

and let

$$\mathrm{E}_{\mathrm{N}_{\lambda_1},\ldots,\mathrm{N}_{\lambda_\mu}}(O(\Omega)) := \bigcup_{U \in \mathrm{D}_{N_{\lambda_1},\ldots,N_{\lambda_\mu}}(\Omega)} U.$$

Consider the sequence

$$\left\{q_{(m_1,\ldots,m_n)}(x,z) = q_{(m_1,\ldots,m_n)}(x_1,\ldots,x_n,z_1,\ldots,z_n) : m_1 = 0,1,2,\ldots,\cdots,m_n = 0,1,2,\ldots\right\}$$

of functions of the $2n$ complex variables $x_1,\ldots,x_n,z_1,\ldots,z_n$, which have the form

$$q_{(m_1,\ldots,m_n)}(x,z) = \sum_{k_1=0}^{m_1} \sigma^{(1)}_{m_1,k_1}(z) \sum_{\nu_1=0}^{k_1} \left(\cdots \left(\sum_{k_n=0}^{m_n} \sigma^{(n)}_{m_n,k_n}(z) \sum_{\nu_n=0}^{k_n} x_1^{\nu_1} z_1^{\nu_1} \ldots x_n^{\nu_n} z_n^{\nu_n} \right) \cdots \right).$$

Suppose ω is the maximal open subset of $\mathbb{C}^{2n}$, into which the functions $q_{(m_1,\ldots,m_n)}(x,z)$ are continuous and the sequence

$$\left\{q_{(m_1,\ldots,m_n)}(x,z) : m_1 = 0,1,2,\ldots,\cdots,m_n = 0,1,2,\ldots\right\}$$

converges compactly to $(1-x_1z_1)^{-1}\ldots(1-x_nz_n)^{-1}$, if $m_{\lambda_1} \to \infty, \ldots, m_{\lambda_\mu} \to \infty$. Set

$$g(\omega,\Omega) = \left\{z = (z_1,\ldots,z_n) \in \mathbb{C}^n : \left(\zeta_1^{-1},\ldots,\zeta_n^{-1},z_1,\ldots,z_n\right) \in \omega,\right.$$

$$\left.\forall\, \zeta_1 \in \overline{\mathbb{C}} - pr_1(\Omega),\ldots,\forall\, \zeta_n \in \overline{\mathbb{C}} - pr_n(\Omega)\right\}.$$

We will show that if Ω is an open polydisk in $\mathbb{C}^n$ or $\Omega = \mathbb{C}^n$, then

$$g(\omega,\Omega) \subset \mathrm{E}_{\mathrm{N}_{\lambda_1},\ldots,\mathrm{N}_{\lambda_\mu}}(O(\Omega)).$$

When $n=1$, our assertion follows directly from Eiermann's theorem. As it is pointed out in [55], the case $n=1$ can be viewed as an extension of Okada's classical result ([114]) in a generalized form which is due to Gawronski and Trautner ([62]): «*if* $\Omega \neq \mathbb{C}$,

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m \geq 0, 0 \leq k \leq m}, \quad \pi_{m,k} \in \mathbb{C},$$

and the sequence

$$\left\{\sum_{k=0}^{m}\pi_{m,k}\sum_{\nu=0}^{k}z^{\nu}: z\in\mathbb{C},\ m=0,1,2,\ldots\right\}$$

converges compactly into an open set H containing 0 *to* $(1-z)^{-1}$, *then*

$$\bigcap_{\zeta\in\mathbb{C}-\Omega}\zeta H\subset \mathbb{E}_{\mathrm{N}_1}(O(\Omega)).»$$

We begin with the following *Proposition*, which lists some of the main properties of the set $g(\omega,\Omega)$:

Proposition 3.1.4. ([37]) **(a).** $g(\omega,\Omega)\subset pr_1(\Omega)\times\cdots\times pr_n(\Omega)$;

(b). $g(\omega,\Omega)$ *is an open subset of* $\mathbb{C}^n$;

(c). $\omega\supset\mathbb{C}^n\times\{0\}$ *if and only if*

$$\lim_{m_{\lambda_1}\to\infty,\ldots,m_{\lambda_\mu}\to\infty}\sum_{k_1=0}^{m_1}\sigma^{(1)}_{m_1,k_1}(0)\cdots\sum_{k_n=0}^{m_n}\sigma^{(n)}_{m_n,k_n}(0)=1\,;$$

(d). *if* $\{\Omega^{(i)}=\Delta^n(0,\delta^{(i)}): i=0,1,2,\ldots\}$ *is a sequence of open polydisks such that* $\Omega^{(i)}\subseteq\Omega^{(i+1)}\ (i\geq 0)$ *and* $\mathbb{C}^n=\bigcup_{i=0}^{\infty}\Omega^{(i)}$,

then

$$g(\omega,\Omega^{(i)})\subset g(\omega,\Omega^{(i+1)})\subset g(\omega,\mathbb{C}^n)\ (i\geq 0)\ \textit{and}\ g(\omega,\mathbb{C}^n)=\bigcup_{i=0}^{\infty}g(\omega,\Omega^{(i)}).$$

Proof. *(a).* Let $z=(z_1,\ldots,z_n)\notin pr_1(\Omega)\times\cdots pr_n(\Omega)$. Suppose $z_j\in\overline{\mathbb{C}}-pr_j(\Omega)$. If $z\in g(\omega,\Omega)$, then

$$\left(\zeta_1^{-1},\ldots,\zeta_n^{-1},z_1,\ldots,z_n\right)\in\omega\,,$$

whenever $\zeta_1\in\overline{\mathbb{C}}-pr_1(\Omega),\ldots,\zeta_n\in\overline{\mathbb{C}}-pr_n(\Omega)$.

Choosing $\zeta_j=z_j$, the convergence

$$\lim_{m_{\lambda_1}\to\infty,\dots,m_{\lambda_\mu}\to\infty} q_{(m_1,\dots,m_n)}\left(\zeta_1^{-1},\dots,\zeta_{j-1}^{-1},z_j^{-1},\zeta_{j+1}^{-1},\dots,\zeta_n^{-1},z_1,\dots,z_n\right)$$
$$=\left(1-\zeta_1^{-1}.z_1\right)^{-1}\dots\left(1-\zeta_{j-1}^{-1}.z_{j-1}\right)^{-1}.\left(1-z_j^{-1}.z_j\right)^{-1}.\left(1-\zeta_{j+1}^{-1}.z_{j+1}\right)^{-1}\dots\left(1-\zeta_n^{-1}.z_n\right)^{-1}$$

leads to a contradiction.

(b). Setting

$$\times_{j=1}^{n}\left[\left(pr_j(\Omega)\right)^c\right]^{-1}=\{(\zeta_1,\dots,\zeta_n)\in\mathbf{C}^n:\zeta_j^{-1}\in\overline{\mathbf{C}}-pr_j(\Omega),\ for\ j=1,2,\dots,n\},$$

one can write

$$g(\omega,\Omega)=\{z\in\mathbf{C}^n:(\zeta,z)\in\omega\ for\ \zeta\in\times_{j=1}^{n}\left[\left(pr_j(\Omega)\right)^c\right]^{-1}\}.$$

Fix $z^{\circ}=(z_1^{\circ},\dots,z_n^{\circ})\in g(\omega,\Omega)$. Since ω is an open set, for every

$$\zeta\in\times_{j=1}^{n}\left[\left(pr_j(\Omega)\right)^c\right]^{-1},$$

there are $\varepsilon_1(\zeta)>0,\dots,\varepsilon_n(\zeta)>0,\delta_1(\zeta)>0,\dots,\delta_n(\zeta)>0$ with

$$(\xi,z)\in\omega,\ if\ \left|\xi_j-\zeta_j\right|<\varepsilon_j(\zeta)\quad and\quad \left|z_j-z_j^{\circ}\right|<\delta_j(\zeta)\qquad (for\ j=1,2,\dots,n).$$

Further, the set

$$\times_{j=1}^{n}\left[\left(pr_j(\Omega)\right)^c\right]^{-1}$$

is closed (because $pr_1(\Omega),\dots,pr_n(\Omega)$ are open) and bounded (because for any $j=1,2,\dots n$ the projection $pr_j(\Omega)$ contains a neighborhood of 0). Thus,

$$\times_{j=1}^{n}\left[\left(pr_j(\Omega)\right)^c\right]^{-1}$$

is compact in $\mathbf{C}^n$, and, consequently, we can choose

$$\xi^{(1)},\dots,\xi^{(N)}\in\times_{j=1}^{n}\left[\left(pr_j(\Omega)\right)^c\right]^{-1}$$

such that

$$\times_{j=1}^{n}\left[\left(pr_j(\Omega)\right)^c\right]^{-1}\subset\bigcup_{i=1}^{N}\Delta^n\left(\xi^{(i)},\varepsilon\left(\xi^{(i)}\right)\right).$$

Defining

$$r_1 = \min\{\delta_1(\xi^{(i)}) : i = 1,2,\dots,N\}, \dots, r_n = \min\{\delta_n(\xi^{(i)}) : i = 1,2,\dots,N\},$$

we obtain

$$(\zeta, z) \in \omega$$

for every $\zeta \in \times_{j=1}^{n}\left[(pr_j(\Omega))^c\right]^{-1}$ and every $z \in \mathbb{C}^n$ with $|\overset{\circ}{z}_1 - z_1| < r_1, \dots, |\overset{\circ}{z}_n - z_n| < r_n$. Thus,

$$\Delta^n\left(\overset{\circ}{z}, (r_1, \dots, r_n)\right) \subset g(\omega, \Omega),$$

and therefore $g(\omega, \Omega)$ is open.

(c). It is an immediate consequence of the fact that

$$\lim_{m_{\lambda_1}\to\infty,\dots,m_{\lambda_\mu}\to\infty} q_{(m_1,\dots,m_n)}(x,0) = (1 - x_1\ 0)^{-1}\dots(1 - x_n\ 0)^{-1}.$$

(d). It is clear that

$$g(\omega, \Omega^{(i)}) \subset g(\omega, \Omega^{(i+1)}) \subset g(\omega, \mathbb{C}^n), \ \forall\ i = 0,1,2,\dots$$

Thus, we have only to show that

$$\bigcup_{i=0}^{\infty} g(\omega, \Omega^{(i)}) = g(\omega, \mathbb{C}^n).$$

From the trivial inclusion $g(\omega, \Omega^{(i)}) \subset g(\omega, \mathbb{C}^n)$, it follows that

$$g(\omega, \mathbb{C}^n) \supset \bigcup_{i=0}^{\infty} g(\omega, \Omega^{(i)}).$$

This means that for every i there exist

$$\zeta_1^{(i)} \in \Delta^1(0, \delta_1^{(i)}), \dots, \zeta_n^{(i)} \in \Delta^1(0, \delta_n^{(i)})$$

such that $\left((\zeta_1^{(i)})^{-1}, \dots, (\zeta_n^{(i)})^{-1}, z_1, \dots, z_n\right) \notin \omega$. Obviously,

$$\lim_{i\to\infty} \zeta_j^{(i)} = \infty$$

for any $j = 1,2,\dots,n,$ and therefore, $z \notin g(\omega, \mathbb{C}^n)$. Hence,

$$g(\omega, \mathbb{C}^n) = \bigcup_{i=0}^{\infty} g(\omega, \Omega^{(i)}),$$

which ends the *Proof* of *Proposition 3.1.4.*

Remarks 3.1.5. (a) Part *(c)* of *Proposition 3.1.4* can be regarded as a generalization of the assumption $0 \in H$ in *Gawronski-Trautner's Theorem*.

(b). A question which may be asked is whether the domain $g(\omega,\Omega)$ is contained in Ω. The following counterexample shows that the answer is, in general, negative. For, choose

$$\Omega = \{(z_1,z_2) \in \mathbb{C}^2 : 0 < |z_1| < |z_2| < 1\} \cup \Delta^2\left((0,0),\left(\frac{1}{2},\frac{1}{2}\right)\right),$$

and

$$\mathrm{M} = (\pi_{m,k})_{m\geq 0, 0\leq k\leq m} = (\delta_{m,k})_{m\geq 0, k\geq 0}$$

where $\delta_{m,k}$ is *the Kronecker symbol*. Let

$$\omega = \{(x_1,x_2,z_1,z_2) \in \mathbb{C}^4 : (x_1,x_2,z_1,z_2) \in \mathbf{F}_{\mathrm{M,M}} \},$$

with

$\mathbf{F}_{\mathrm{M,M}} = \bigcup_{Q \in \mathbf{Q}_{M,M}} Q,$

and

$\mathbf{Q}_{\mathrm{M,M}} = \{Q \subset \mathbb{C}^2$:*the* (M,M)-*transform of the sequence of partial sums of the geometric series*

$$\sum_{\nu_1,\nu_2=0}^{\infty} z_1^{\nu_1} . z_2^{\nu_2}$$

converges to $(1-z_1)^{-1}(1-z_2)^{-1}$ *compactly on Q, if* $m \to \infty$ }.

Evidently, $0 \in \Omega$, $0 \in \mathbf{F}_{\mathrm{M,M}}$ but

$$g(\omega,\Omega) = \Delta^2(0,1) \not\subset \Omega.$$

For our purposes, we shall need the following *Lemma*.

Lemma 3.1.6. *Let* $\Delta^n(0,r) = \Delta^n((0,0,...,0),(r_1,r_2,...,r_n))$ *be an open polydisk in* $\mathbb{C}^n$, *with center at* 0. *If* K *is a compact subset of* $\Delta^n(0,r)$, *one can find another open polydisk*

$\Delta^n(0,\rho)=\Delta^n((0,0,...,0),(\rho_1,\rho_2,...,\rho_n))$, *such that*

$$K \subset\subset \Delta^n(0,\rho) \subset\subset \Delta^n(0,r).$$

Suppose that the functions $q_{(m_1,...,m_n)}(x,z)$ *are continuous in the set*

$$\left[b\Delta^n(0,\rho)\right]^{-1} \times K = \left\{(\zeta_1,\zeta_2,...,\zeta_n) \in \mathbb{C} : \left|\zeta_j\right| = \rho_j^{-1}, \, for \; j=1,2,...,n\right\} \times K.$$

If $f \in O(\Delta^n(0,r))$, *then for all* $(m_1,m_2,...,m_n)$ *there holds*

$$\sup_{z\in K}\left|f(z)-T_f\left(q_{(m_1,...,m_n)}(x,z)\right)\right|$$

$$\leq L(f,\rho)\sup_{(t,z)\in[b\Delta^n(0;\rho)]^{-1}\times K}\left|(1-t_1 z_1)^{-1}...(1-t_n z_n)^{-1}-q_{(m_1,...,m_n)}(t,z)\right|,$$

where $L(f,\rho)$ *is a positive constant depending only on* f *and* ρ.

Proof. Let $z \in K$. By applying *Corollary 3.1.2* and *Cauchy's Integral Formula* for polydisks, we obtain

$$\left|f(z)-T_f\left(q_{(m_1,...,m_n)}(x,z)\right)\right| = \left|T_f\left((1-x_1z_1)^{-1}...(1-x_nz_n)^{-1}-q_{(m_1,...,m_n)}(x,z)\right)\right|$$

$$=\left|T_f\left((2\pi i)^{-n}.\int_{s\in[b\Delta^n(0;\rho)]^{-1}}\frac{(1-s_1z_1)^{-1}...(1-s_nz_n)^{-1}-q_{(m_1,...,m_n)}(s,z)}{(s_1-x_1)...(s_n-x_n)}ds_1...ds_n\right)\right|$$

$$=\left|(2\pi i)^{-n}.\int_{s\in[b\Delta^n(0;\rho)]^{-1}}\left[T_f\frac{s_1^{-1}...s_n^{-1}}{(1-x_1.s_1^{-1})...(1-x_n.s_n^{-1})}\right].\left[(1-s_1.z_1)^{-1}(1-s_n.z_n)^{-1}\right.\right.$$

$$\left.\left.-q_{(m_1,...,m_n)}(s,z)\right]ds_1...ds_n\right|$$

$$=\left|(2\pi i)^{-n}.\int_{s\in[b\Delta^n(0;\rho)]^{-1}}\left[s_1^{-1}...s_n^{-1}.f(s_1^{-1},...,s_n^{-1})\right]\left[(1-s_1.z_1)^{-1}...(1-s_n.z_n)^{-1}-q_{(m_1,...,m_n)}(s,z)\right]ds_1...ds_n\right|$$

$$\leq L(f,\rho)\sup_{(t,z)\in[b\Delta^n(0;\rho)]^{-1}\times K}\left|(1-t_1 z_1)^{-1}...(1-t_n z_n)^{-1}-q_{(m_1,...,m_n)}(t,z)\right|,$$

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which completes the *Proof* of the *Lemma*.

The following *Theorem* is a consequence of *Lemma 3.1.6*, but is more useful since the choice of the compact K and of the polydisk $\Delta^n(0,\rho)$ is eliminated.

Theorem 3.1.7. *If* $\Omega=\Delta^n(0,r)$ *or* $\Omega=\mathbb{C}^n$, *then*

$$g(\omega,\Omega)\subseteq \mathbb{E}_{\mathrm{N}_{\lambda_1},\dots,\mathrm{N}_{\lambda_\mu}}(O(\Omega)).$$

Proof. First, we assume that $\Omega=\Delta^n(0,r)$. It is sufficient to prove the following assertion:

for every $\mathrm{z}^\circ=(\mathrm{z}_1^\circ,\dots,\mathrm{z}_n^\circ)\in g(\omega,\Delta^n(0,r))$, *there exists a closed polydisk* $\overline{\Delta^n(z^\circ,(r_1^\circ,\dots,r_n^\circ))}\subset g(\omega,\Delta^n(0,r))$ *satisfying* $r_j^\circ\neq 0\ \forall\ j=1,2,\dots,n$ *and*

$$\lim_{m_{\lambda_1}\to\infty,\dots,m_{\lambda_\mu}\to\infty} T_f\left(q_{(m_1,\dots,m_n)}(x,z)\right)=f(z)$$

uniformly on $\overline{\Delta^n(z^\circ,(r_1^\circ,\dots,r_n^\circ))}$, *whenever* $f\in O(\Delta^n(0,r))$.

We must distinguish two cases: $\mathrm{z}^\circ\neq 0$ and $\mathrm{z}^\circ=0$.

<u>*1st case : $z^\circ\neq 0$*</u>. Since $g(\omega,\Delta^n(0,r))$ is open (*Proposition 3.1.4.(b)*), one can find $r^\circ=(r_1^\circ,\dots,r_n^\circ)\in\mathbb{R}_+^n$ such that $\overline{\Delta^n(z^\circ,r^\circ)}\subset g(\omega,\Delta^n(0,r))$. By the definition of $g(\omega,\Delta^n(0,r))$, the compact set

$$\{(x,z)\in\mathbb{C}^{2n}: x\in \times_{j=1}^n\left[\overline{\mathbb{C}}-\Delta^1(0,r_j)\right]^{-1} \textit{and } z\in\overline{\Delta^n(z^\circ,r^\circ)}\}$$

is contained in the open set ω. Therefore, there exist $\varepsilon_1>0,\dots,\varepsilon_n>0$ such that

$$\{(x,z)\in\mathbb{C}^{2n}: x\in \times_{j=1}^n\left[\overline{\mathbb{C}}-\Delta^1(0,r_j)\right]^{-1} \textit{and } z\in\overline{\Delta^n(z^\circ,r^\circ)}\}\subset\omega.$$

Since $\Delta^n(z^\circ,r^\circ)\subset g(\omega,\Delta^n(0,r))\subset\Delta^n(0,r)$ (*Proposition 3.1.4.(a)*), we obtain

$$dist\left(\left[u_1 \in \Delta^1(0,r_1): |u_1| = \max\{r_1 - \varepsilon_1, \sup_{z=(z_1,\dots,z_n)\in\overline{\Delta^n(z^\circ;r^\circ)}} |z_1|\}\right], \vartheta\Delta^1(0,r_1)\right) = \varepsilon_1^{'} > 0,$$

………

$$dist\left(\left[u_n \in \Delta^1(0,r_n): |u_n| = \max\{r_n - \varepsilon_n, \sup_{z=(z_1,\dots,z_n)\in\overline{\Delta^n(z^\circ;r^\circ)}} |z_n|\}\right], \vartheta\Delta^1(0,r_n)\right) = \varepsilon_n^{'} > 0.$$

Defining

$$\rho_1 = r_1 - \varepsilon_1^{'}, \dots, \rho_n = r_n - \varepsilon_n^{'},$$

we see that *Lemma 3.1.6* can be applied to $\Delta^n(0,\rho) = \Delta^n((0,0,\dots,0),(\rho_1,\rho_2,\dots,\rho_n))$ and the compact set $\overline{\Delta^n(z^\circ,r^\circ)}$ and, thus our assertion follows.

2

2^{nd} *case*: $z^\circ=0$. Choose $\rho = (\rho_1,\dots,\rho_n)$ such that $\overline{\Delta^n(0,\rho)} \subset \Delta^n(0,r)$. By assumption, the compact set $\{(x,0) \in \mathbb{C}^{2n}: x \in \overline{\Delta^n(0,\rho)}\}$ is contained in the open set ω. Hence, $\{(x,z) \in \mathbb{C}^{2n}: x \in \overline{\Delta^n(0,\rho)}, z \in \Delta^n(0,\tau)\} \subset \omega$ for a suitably chosen small polydisk $\Delta^n(0,\tau)$. It is obvious that *Lemma 3.1.6* can be applied to the polydisk $\Delta^n(0,\rho)$ and its compact subset $\overline{\Delta^n(z^\circ,r^\circ)} = \overline{\Delta^n(z^\circ,(r_1^\circ,\dots,r_n^\circ))}$ where $0<r_1^\circ<\min\{\rho_1,\tau_1\},\dots,0<r_n^\circ<\min\{\rho_n,\tau_n\}$ and our assertion is again proved.

We conclude that

$$g(\omega,\Delta^n(0,r)) \subset \mathrm{E}_{\mathrm{N}_{\lambda 1},\dots,\mathrm{N}_{\lambda\mu}}(O(\Delta^n(0,r))).$$

Next , suppose $\Omega=\mathbb{C}^n$. As before, it is enough to show that

«*for every* $z^\circ = (z_1^\circ,\dots,z_n^\circ) \in g(\omega,\mathbb{C}^n)$ *there is a closed polydisk* $\overline{\Delta^n(z^\circ,(r_1^\circ,\dots,r_n^\circ))} \subset g(\omega,\mathbb{C}^n)$, *satisfying* $r_j \neq 0 \quad \forall j=1,\dots,n$ *and*

$$\lim_{m_{\lambda_1}\to\infty,\dots,m_{\lambda_\mu}\to\infty} T_f\left(q_{(m_1,\dots,m_n)}(x,z)\right) = f(z)$$

uniformly on $\overline{\Delta^n(z^\circ,(r_1^\circ,\dots,r_n^\circ))}$, *whenever* $f \in O(\mathbb{C}^n)$».

Let $z^{\circ} = (z_1^{\circ},\ldots,z_n^{\circ}) \in g(\omega, \mathbb{C}^n)$ and let $f \in O$ $(\mathbb{C}^n)$. Choose a sequence $\{\Delta^n(0,\delta^{(i)}) : i = 0,1,2,\ldots\}$ a sequence of open polydisks in $\mathbb{C}^n$ such that

$$\Delta^n(0,\delta^{(i)}) \subseteq \Delta^n(0,\delta^{(i+1)}) \text{ and } \mathbb{C}^n = \bigcup_{i=0}^{\infty} \Delta^n(0,\delta^{(i)}) .$$

By *Proposition 3.1.4.(d)*,

$$g(\omega, \Delta^n(0,\delta^{(i)})) \subseteq g(\omega, \Delta^n(0,\delta^{(i+1)})) \quad \text{and} \quad g(\omega, \mathbb{C}^n) = \bigcup_{i=0}^{\infty} g(\omega, \Delta^n(0,\delta^{(i)})) .$$

Thus, we can find i_0, such that $z^{\circ} \in g(\omega, \Delta^n(0,\delta^{(i_\circ)}))$. Since $g(\omega, \Delta^n(0,\delta^{(i_\circ)}))$ is an open set in $\mathbb{C}^n$ (*Proposition 3.1.4.(b)*), there exist $r_1^{\circ} \neq 0,\ldots, r_n^{\circ} \neq 0$ with

$$\overline{\Delta^n(z^{\circ},(r_1^{\circ},\ldots,r_n^{\circ}))} \subset g(\omega, \Delta^n(0,\delta^{(i_\circ)})) .$$

On the other hand $f \in O$ $(\mathbb{C}^n)$. Hence $f \in O$ $(\Delta^n(0,\delta^{(i_\circ)}))$. Evidently, the first part of this *Theorem* can be applied to the polydisk $\Delta^n(0,\delta^{(i_\circ)})$ and the compact set $\overline{\Delta^n(z^{\circ},(r_1^{\circ},\ldots,r_n^{\circ}))}$ to obtain

$$\lim_{m_{\lambda_1}\to\infty,\ldots,m_{\lambda_\mu}\to\infty} T_f\left(q_{(m_1,\ldots,m_n)}(x,z)\right) = f(z)$$

uniformly on $\overline{\Delta^n(z^{\circ},(r_1^{\circ},\ldots,r_n^{\circ}))}$. This implies that

$$g(\omega, \mathbb{C}^n) \subset \mathrm{E}_{N_{\lambda_1},\ldots,N_{\lambda_\mu}} (O\,(\mathbb{C}^n))$$

and *Proof* of the *Theorem* is complete.

Corollary 3.1.8. ([38]) *Let f be a complex function analytic in an open polydisk* $\Delta^n(0,r)$. *Let ω be the maximal open neighborhood of* $\Delta^n(0,(r_1^{-1},\ldots,r_n^{-1})) \times \Delta^n(0,r)$ *into which the series*

$$\sum_{\nu_1,\ldots,\nu_n=0}^{\infty} x_1^{\nu_1} z_1^{\nu_1} \ldots x_n^{\nu_n} z_n^{\nu_n} \qquad \left(|x_j z_j| < 1,\ j = 1,2,\ldots,n\right)$$

converges compactly to $(1 - x_1 z_1)^{-1} \ldots (1 - x_n z_n)^{-1}$.

If the generating polynomials

$$V_{(m_1+1,\dots,m_n+1)}(x_1,\dots,x_n)=\gamma\prod_{j=1}^{n}V_{m_j+1}(x_j)$$

of a Padé-type approximation satisfy

$$(C_n)\qquad \lim_{m_{\lambda_1}\to\infty,\dots,m_{\lambda_\mu}\to\infty}\left(1-\frac{V_{m_1+1}(x_1)}{V_{m_1+1}(z_1^{-1})}\right)\dots\left(1-\frac{V_{m_n+1}(x_n)}{V_{m_n+1}(z_n^{-1})}\right)=1$$

compactly in ω, *then , for any* $f\in O(\Delta^n(0,r))$, *there holds*

$$\lim_{m_{\lambda_1}\to\infty,\dots,m_{\lambda_\mu}\to\infty}(m_1,\dots m_n/m_1+1,\dots,m_n+1)_f(z)=f(z)$$

compactly in $\Delta^n(0,r)$.

Proof. It suffices to see that condition (C_n) implies that

$$\lim_{m_{\lambda_1}\to\infty,\dots,m_{\lambda_\mu}\to\infty}q_{(m_1,\dots,m_n)}(x,z_1)=(1-x_1\ z_1)^{-1}\dots(1-x_n\ z_n)^{-1}$$

and then apply *Theorem 3.1.7*.

Corollary 3.1.9.([38]) *Let us consider the sequence of generating polynomials*

$$\left\{V_{(m_1+1,\dots,m_n+1)}(x_1,\dots,x_n)=\gamma\prod_{j=1}^{n}(x_j-\beta_j)^{m_j+1}:\gamma\in\mathbb{C}-\{0\},\beta_j\in\mathbb{C},m_j=0,1,\dots(j=1,\dots,n)\right\}$$

If $f\in O(\Delta^n(0,r))$, *then the corresponding sequence of Padé-type approximants to* f *converges to* f *compactly in some open subset of* $\Delta^n(0,r)$, *more precisely*

$$\lim_{m_1\to\infty,\dots,m_n\to\infty}(m_1,\dots m_n/m_1+1,\dots,m_n+1)_f(z)=f(z)$$

uniformly on every compact subset of

$$\left\{z=(z_1,\dots,z_n)\in\Delta^n(0,r):\left|z_j^{-1}-\beta_j\right|>\sup_{|\zeta_j|>r_j}\left|\zeta_j^{-1}-\beta_j\right|,\ for\ j=1,2,\dots,n\right\}.$$

Corollary 3.1.10 ([38]) *Let us consider the sequence of generating polynomials*

$$\{V_{(m_1+1,\dots,m_n+1)}(x_1,\dots,x_n)=\gamma\prod_{i=1}^{n}\prod_{j=0}^{m_j}\left(x_i-\beta_i^{(j)}\right):\gamma\in\mathbb{C}-\{0\},\ \beta_i^{(j)}\in\mathbb{C},$$

$$m_i=0,1,\dots\ (1\le i,j\le n)\}$$

Let also $f\in O\left(\Delta^n(0,r)\right)$. *If*

$$\lim_{j\to\infty}\beta_i^{(j\,k_i+s)}=\gamma_i^{(s)},\ s=0,1,\dots,k_i-1\ \ (i=1,2,\dots,n),$$

then

$$\lim_{m_1\to\infty,\dots,m_n\to\infty}(m_1,\dots m_n/m_1+1,\dots,m_n+1)_f(z)=f(z)$$

compactly on

$$\times_{i=1}^{n}\{z\in\mathbb{C}:z_i^{-1}\notin\overline{L(\rho_i^{\circ})}\ \}\cap\Delta^n(0,r),$$

where we have used the notations

$$L(\rho_i):=\{z_i\in\mathbb{C}:\left|\prod_{j=0}^{k_i-1}\left(z_i-\gamma_i^{(j)}\right)\right|<\rho_i\},\ \ \rho_i>0$$

and

$$\rho_i^{\circ}:=\sup_{|\zeta_i|>r_i}\left|\prod_{j=0}^{k_i-1}\left(\zeta_i^{-1}-\gamma_i^{(j)}\right)\right|.$$

Proof. From *Theorem 3.1.7*, it follows that

$$\lim_{m_1\to\infty,\dots,m_n\to\infty}(m_1,\dots m_n/m_1+1,\dots,m_n+1)_f(z)=f(z)$$

uniformly on every compact subset of

$$\{z=(z_1,\dots,z_n)\in\Delta^n(0,r):$$

$$\lim_{m_1\to\infty,\dots,m_n\to\infty}\left[\sum_{i=1}^{n}\frac{\prod_{j=0}^{m_i}\left(\xi_i^{-1}-\beta_i^{(j)}\right)}{\prod_{j=0}^{m_i}\left(z_i^{-1}-\beta_i^{(j)}\right)}-\sum_{\substack{i,k=1\\(i\neq k)}}^{n}\frac{\prod_{j=0}^{m_i}\left(\xi_i^{-1}-\beta_i^{(j)}\right)\prod_{j=0}^{m_k}\left(\xi_k^{-1}-\beta_k^{(j)}\right)}{\prod_{j=0}^{m_i}\left(z_i^{-1}-\beta_i^{(j)}\right)\prod_{j=0}^{m_k}\left(z_k^{-1}-\beta_k^{(j)}\right)}+\cdots\right.$$

$$\left.\cdots+(-1)^{n+1}\frac{\prod_{i=0}^{n}\prod_{j=0}^{m_i}\left(\xi_i^{-1}-\beta_i^{(j)}\right)}{\prod_{i=0}^{n}\prod_{j=0}^{m_i}\left(z_j^{-1}-\beta_i^{(j)}\right)}\right],\ whenever\ |\xi_1|\geq r_1,\dots,|\xi_n|\geq r_n\}$$

$$\supset$$

$$\left\{z=(z_1,\dots,z_n)\in\Delta^n(0,r):\left|\frac{\prod_{j=0}^{k_i-1}\left(\xi_i^{-1}-\beta_i^{(j)}\right)}{\prod_{j=0}^{k_i-1}\left(z_i^{-1}-\beta_i^{(j)}\right)}\right|<1\ ,\ for\ i=1,2,\dots,n\ \ whenever\ |\xi_1|\geq r_1,\dots,|\xi_n|\geq r_n\right\}$$

$$=$$

$$\left\{z=(z_1,\dots,z_n)\in\Delta^n(0,r):\left|\frac{\prod_{j=0}^{k_i-1}\left(\xi_i^{-1}-\beta_i^{(j)}\right)}{\prod_{j=0}^{k_i-1}\left(z_i^{-1}-\beta_i^{(j)}\right)}\right|<1\ ,\ for\ i=1,2,\dots,n\ \ whenever\ |\xi_1|\geq r_1,\dots,|\xi_n|\geq r_n\right\}=$$

$$\left\{z=(z_1,\dots,z_n)\in\Delta^n(0,r):\left[\frac{\sup_{|\xi_i|\geq r_i}\prod_{j=0}^{k_i-1}\left(\xi_i^{-1}-\gamma_i^{(j)}\right)}{\prod_{j=0}^{k_i-1}\left(z_i^{-1}-\gamma_i^{(j)}\right)}\right]\leq 1,\ for\ i=1,2,\dots,n\right\}$$

$$=$$

$$\left\{z=(z_1,\dots,z_n)\in\Delta^n(0,r):\left|\prod_{j=0}^{k_i-1}\left(z_i^{-1}-\gamma_i^{(j)}\right)\right|\geq\sup_{|\xi_i|\geq r_i}\left|\prod_{j=0}^{k_i-1}\left(\xi_i^{-1}-\gamma_i^{(j)}\right)\right|,\ for\ i=1,2,\dots,n\right\}=$$

$$\left\{z=(z_1,\dots,z_n)\in\Delta^n(0,r):z_i^{-1}\notin\overline{L\left(\rho_i^{\circ}\right)},\ for\ i=1,2,\dots,n\right\},$$

this completes the *Proof.*

Corollary 3.1.11. ([38]) *Assume that the generating polynomials have the form*

$$V_{(m_1+1,\dots,m_n+1)}(x_1,\dots,x_n)=\gamma\prod_{j=1}^{n}C_{m_j+1}(x_j)\quad (\gamma\in\mathbb{C}-\{0\}, m_1=0,1,\dots,\cdots,m_n=0,1,\dots)$$

where

$$\mathrm{TCH}_{m_{j+1}}(x_j)=\cos(m_j\, Arc\cos x_j)=\prod_{i=0}^{m_j}(x_j-\beta_{m_j,i})$$

are the Tchebycheff polynomials, and

$$\beta_{m_j,i}=\cos\left(\frac{2i+1}{2.(m_j+1)}\pi\right)$$

are the zeros of Tchebycheff's polynomials (for $i=0,1,\dots,m_j$*).*

If $f\in O\,(\Delta^n(0,r))$, *then*

$$\lim_{m_1\to\infty,\dots,m_n\to\infty}(m_1,\dots m_n/m_1+1,\dots,m_n+1)_f(z)=f(z)$$

compactly on

$$\{z=(z_1,\dots,z_n)\in\Delta^n(0,r):$$
$$\left(\left|z_j^{-1}-1\right|+\left|z_j^{-1}+1\right|\right)>\sup_{|\xi_j|\geq r_j}\left(\left|\xi_j^{-1}-1\right|+\left|\xi_j^{-1}+1\right|\right),\ \ for\ j=1,2,\dots,n\}.$$

Proof. By *Theorem 3.1.7*, we have

$$\lim_{m_1\to\infty,\dots,m_n\to\infty}(m_1,\dots,m_n/m_1+1,\dots,m_n+1)_f(z)=f(z)$$

uniformly on every compact subset of

$$\Delta^n(0,r)\cap\times_{j=1}^{n}\{z_j\in\mathbb{C}:$$

$$\left| \frac{\prod_{i=0}^{m_j} \left[\xi_j^{-1} - \cos\left(\frac{2i+1}{2(m_j+1)} \pi \right) \right]}{\prod_{i=0}^{m_j} \left[z_j^{-1} - \cos\left(\frac{2i+1}{2.(m_j+1)} \pi \right) \right]} \right| < 1 \, , \left|\xi_j\right| \geq r_j \ and \ m_j = 1,2,... \}$$

$$\supset$$

$\Delta^n(0,r) \cap \times_{j=1}^n \{ z_j \in \mathbb{C}:$

$$\sup_{|\xi_j| \geq r_j} \left| \xi_j^{-1} - \cos\left(\frac{2i+1}{2(m_j+1)} \pi \right) \right| < \left| z_j^{-1} - \cos\left(\frac{2i+1}{2(m_j+1)} \pi \right) \right| , for \ m_j \geq 0 \ and \ i = 1,2,...,n \}.$$

The desired conclusion follows now from the relations

$$\sup_{|\xi_j| \geq r_j} \left| \xi_j^{-1} - \cos\left(\frac{2i+1}{2(m_j+1)} \pi \right) \right| < \sup_{|\xi_j| \geq r_j} \left(\left|\xi_j^{-1} - 1\right| + \left|\xi_j^{-1} + 1\right| \right),$$

and

$$\left| z_j^{-1} - \cos\left(\frac{2i+1}{2(m_j+1)} \pi \right) \right| < \left(\left|z_j^{-1} - 1\right| + \left|z_j^{-1} + 1\right| \right).$$

A natural question which now arises is whether the inclusion

$$g(\omega, \Omega) \subset \mathbb{E}_{\mathrm{N}_{\lambda_1},...,\mathrm{N}_{\lambda_\mu}} (O(\Omega))$$

remains true into an arbitrary open set $\Omega \subset \mathbb{C}^n$ ($0 \in \Omega$, n>1). We shall give two examples showing that the answer is, in general, negative ([39]).

Let Ω be an open subset of $\mathbb{C}^n$, $0 \in \Omega$, such that

$\left(\Omega, \Delta^n(0,1)\right)$ is not a Runge pair and $pr_j(\Omega) = \Delta^1(0,1)$, for $j = 1,2,...,n$.

We will show that there are n infinite triangular matrices $\mathrm{N}_1(z) = \cdots = \mathrm{N}_n(z)$ satisfying

$$g(\omega,\Omega)\not\subset \mathrm{E}_{\mathrm{N}_{\lambda_1},\ldots,\mathrm{N}_{\lambda_\mu}}(O(\Omega)).$$

Let $f\in O(\Omega)$ and let

$$\sum_{\nu_1,\ldots,\nu_n=0}^{\infty} a_{\nu_1,\ldots,\nu_n}^{(f)} w_1^{\nu_1}\ldots w_n^{\nu_n}$$

be the power series expansion of f, around the origin. Choosing,

$$\mathrm{N}_1(z)=\cdots=\mathrm{N}_n(z)=(\delta_{m,k})_{m\ge 0,k\ge 0}$$

($\delta_{m,k}$ is the *Krönecker symbol*), it is obvious that

$$g(\omega,\Omega)=\Omega.$$

If we assume that

$$g(\omega,\Omega)\subset \mathrm{E}_{\mathrm{N}_{\lambda_1},\ldots,\mathrm{N}_{\lambda_\mu}}(O(\Omega)),$$

then we directly get

$$\mathrm{E}_{\mathrm{N}_{\lambda_1},\ldots,\mathrm{N}_{\lambda_\mu}}(O(\Omega))=\Omega.$$

Consequently, it holds

$$\lim_{m_{\lambda_1}\to\infty,\ldots,m_{\lambda_\mu}\to\infty}\sum_{k_1=0}^{m_1}\delta_{m_1,k_1}\sum_{\nu_1=0}^{k_1}\left(\cdots\left(\sum_{k_n=0}^{m_n}\delta_{m_n,k_n}\sum_{\nu_n=0}^{k_n}a_{\nu_1,\ldots,\nu_n}^{(f)}z_1^{\nu_1}\ldots z_n^{\nu_n}\right)\cdots\right)=f(z_1,\ldots,z_n)$$

compactly on Ω and therefore, f is the limit of a sequence of analytic functions in $\Delta^n(0,1)$. Hence, $(\Omega,\Delta^n(0,1))$ is a Runge pair, which is in direct contrast with the imposed hypothesis. We conclude that the assumption

$$g(\omega,\Omega)\subset \mathrm{E}_{\mathrm{N}_{\lambda_1},\ldots,\mathrm{N}_{\lambda_\mu}}(O(\Omega))$$

is wrong and thus, we have proved the

Proposition 3.1.12. *Let* Ω *be an open subset of* $\mathbb{C}^n$($0\in\Omega$), *such that*

$$pr_j(\Omega) = \Delta^1(0,1) \textit{ for } j = 1,2,\ldots,n$$

and $(\Omega, \Delta^n(0,1))$ *is not a Runge pair. Then, there are* n *infinite triangular matrices* $\mathrm{N}_1(z) = \cdots = \mathrm{N}_n(z)$, *such that*

$$g(\omega,\Omega) \not\subset \mathrm{E}_{\mathrm{N}_{\lambda_1},\ldots,\mathrm{N}_{\lambda_\mu}}(O(\Omega)).$$

Corollary 3.1.13. *Let* Ω *be the domain of* $\mathbb{C}^2$ *defined by*

$$\Omega = \Delta^2(0,1) - \{(z_1, z_2) \in \mathbb{C}^2 : z_1 + z_2 = 1\}.$$

Then there are two infinite triangular matrices $\mathrm{N}_1(z)$ *and* $\mathrm{N}_2(z)$ *with*

$$g(\omega,\Omega) \not\subset \mathrm{E}_{\mathrm{N}_1,\mathrm{N}_2}(O(\Omega)).$$

Proof. It suffices to show that Ω satisfies the presuppositions of *Proposition 3.1.12*. Obviously, $0 \in \Omega$ and $pr_1(\Omega) = pr_2(\Omega) = \Delta^1(0,1)$. Further, the open set Ω is a domain of holomorphy. In fact, it is enough to see that $\Delta^2(0,1)$ is a domain of holomorphy of $\mathbb{C}^2$ and that $\{(z_1, z_2) \in \mathbb{C}^2 : z_1 + z_2 = 1\}$ is a hypersurface in $\mathbb{C}^2$. In order to show that $(\Omega, \Delta^n(0,1))$ is not a Runge pair, it is enough to find a set $\mathrm{K} \subset \Omega$ such that $\hat{\mathrm{K}}_{O(\Delta^2(0;1))} \cap \Omega$ is not compact in Ω. If we choose

$$\mathrm{K} = \left\{ \left(\frac{1}{2}, \frac{3}{4} e^{i\theta} \right) : 0 \le \theta \le 2\pi \right\},$$

then $\left(\frac{1}{2}, \frac{1}{2} \right) \in \hat{\mathrm{K}}_{O(\Delta^2(0;1))}$. Since $\left(\frac{1}{2}, \frac{1}{2} \right) \in \vartheta\,\Omega$, the *Proof* is complete.

Next, assume that Ω is the open subset of $\mathbb{C}^2$ defined by

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$$\Omega=\{(z_1,z_2)\in\mathbb{C}^2: |z_1|<1, |z_2|<1, |z_1+z_2|<1\}.$$

It is clear that $0\in\Omega$ and that $pr_1(\Omega)=pr_2(\Omega)=\Delta^1(0,1)$. Moreover, it is easily verified that $(\Omega,\Delta^n(0,1))$ is a Runge pair. We shall show that there are two infinite triangular matrices $\mathrm{N}_1(z)$ and $\mathrm{N}_2(z)$ satisfying

$g(\omega,\Omega)\not\subset \mathrm{E}_{\mathrm{N}_1,\mathrm{N}_2}(O(\Omega))$.

The *Proof* is due to Professor G. Coeuré (private communication).

Choose

$$f:\Omega\to\mathbb{C}:(z_1,z_2)\mapsto f(z_1,z_2)=\frac{1}{1-(z_1+z_2)}\in O(\Omega).$$

Evidently, f can be expressed, in a neighborhood of 0, as

$$f(w_1,w_2)=\sum_{\nu=0}^{\infty}(w_1+w_2)^{\nu}=\sum_{\nu=0}^{\infty}\left(\sum_{p=0}^{\infty}C_\nu^p w_1^{\nu-p}w_2^p\right)=\sum_{q,p=0}^{\infty}C_{q+p}^p w_1^q w_2^p ,$$

where $q=\nu-p$ and $C_\nu^p=\binom{\nu}{p}$. For any $(w_1,w_2)\in\Omega$, set

$$S_k(w_1,w_2)=\sum_{q,p=0}^{\infty}C_{q+p}^p w_1^q w_2^p \quad (k=0,1,2,...).$$

Let us study the difference

$$S_{k+1}(w_1,w_2)-S_k(w_1,w_2),\text{ for }(w_1,w_2)\in\Omega.$$

Suppose (z_1,z_2) is a point of Ω. We have

$$S_{k+1}(z_1,z_2)-S_k(z_1,z_2)=\sum_{p=0}^{k}C_{k+1+p}^p z_1^{k+1}z_2^p+\sum_{q=0}^{k}C_{q+k+1}^{k+1}z_1^q z_2^{k+1}+C_{2k+2}^{k+1}z_1^{k+1}z_2^{k+1}.$$

In particular, when $z_2=-z_1$ and $k+1=2k'$ (i.e., even), the above difference becomes

$$S_{k+1}(z_1,z_2)-S_k(z_1,z_2)=S_{2k'}(z_1,-z_1)-S_{2k'-1}(z_1,-z_1)$$

$$= \sum_{p=0}^{2k'-1} C_{2k'+p}^{p} (-1)^p z_1^{2k'+p} + \sum_{q=0}^{2k'-1} C_{2k'+q}^{2k'} z_1^{2k'+q} + C_{4k'}^{2k'} z_1^{4k'} .$$

If we restrain our attention to $z_1 > 0$, then it is easily seen that

$$\sum_{p=0}^{2k'-1} C_{2k'+p}^{p} (-1)^p z_1^{2k'+p} + \sum_{q=0}^{2k'-1} C_{2k'+q}^{2k'} z_1^{2k'+q} > 0 \quad \text{and} \quad C_{4k'}^{2k'} z_1^{4k'} > 0 .$$

Assuming that

$$\lim_{k\to\infty} S_k(w_1, w_2) = \frac{1}{1-(w_1+w_2)} \text{, whenever } (w_1, w_2) \in \Omega ,$$

we obtain

$$\lim_{k\leftarrow\infty} (S_{k+1}(w_1, w_2) - S_k(w_1, w_2)) = 0 \quad \text{for any } (w_1, w_2) \in \Omega .$$

Consequently,

$$\lim_{k'\to\infty} C_{4k'}^{2k'} z_1^{4k'} = 0 .$$

Since

$$C_{4k'}^{2k'} = \frac{(4k')!}{[(2k')!]^2} ,$$

it follows, from *Stirling's Formula*, that

$$\lim_{k'\to\infty} C_{4k'}^{2k'} z_1^{4k'} = \lim_{k'\to\infty} \left[\frac{\left(4k'/e\right)^{4k'} \sqrt{8\pi k'}}{\left(2k'/e\right)^{4k'} \left(\sqrt{4\pi k'}\right)^2} z_1^{4k'} \right] = \lim_{k'\to\infty} \frac{2^{4k'} z_1^{4k'}}{\sqrt{2\pi k'}} \neq 0$$

for a $z_1 > 0$, suitably chosen near to 1. This shows that the assumption

$$\lim_{k\to\infty} S_k(w_1, w_2) = [1-(w_1+w_2)]^{-1} \text{ whenever } (w_1, w_2) \in \Omega$$

is erroneous. Hence, the point $(z_1, -z_1) \in \Omega$ (with $z_1 > 0$, z_1 near to 1) satisfies

$$f(z_1, -z_1) \neq \sum_{\nu_1,\nu_2=0}^{\infty} a_{\nu_1,\nu_2}^{(f)} z_1^{\nu_1} (-z_1)^{\nu_2} .$$

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Choosing

$$\mathrm{N}_1(z)=\mathrm{N}_2(z)=(\delta_{m,k})_{m\geq 0,k\geq 0}$$

($\delta_{m,k}$ is again the *Krönecker symbol*), it is readily seen that

$$\lim_{m\to\infty,m_2\to\infty}\sum_{k_1=0}^{m_1}\delta_{m_1,k_1}\sum_{\nu_1=0}^{k_1}\left(\sum_{k_2=0}^{m_2}\delta_{m_2,k_2}\sum_{\nu_2=0}^{k_2}a_{\nu_1,\nu_2}^{(f)}z_1^{\nu_1}(-z_1)^{\nu_2}\right)=\sum_{\nu_1,\nu_2=0}^{\infty}a_{\nu_1,\nu_2}^{(f)}z_1^{\nu_1}(-z_1)^{\nu_2}.$$

This implies that

$$(z_1,-z_1)\notin \mathrm{E}_{\mathrm{N}_1,\mathrm{N}_2}(O(\Omega)).$$

But, on the other hand, there holds

$$(z_1,-z_1)\in g(\omega,\Omega)=\Omega,$$

and hence we have proved the following

Proposition 3.1.14. *Let* Ω *be the domain of* $\mathbb{C}^2$ *defined by*

$$\Omega=\{(z_1,z_2)\in\mathbb{C}^2:|z_1|<1,|z_2|<1,|z_1+z_2|<1\}.$$

There are two infinite triangular matrices $\mathrm{N}_1(z)$ *and* $\mathrm{N}_2(z)$ *fulfilling*

$$g(\omega,\Omega)\not\subset \mathrm{E}_{\mathrm{N}_1,\mathrm{N}_2}(O(\Omega)).$$

We shall now modify the form of the set $g(\omega,\Omega)$ and construct a new domain $G(\omega,\Omega)$ always contained in $\mathrm{E}_{\mathrm{N}_{\lambda_1},\ldots,\mathrm{N}_{\lambda_\mu}}(O(\Omega))$, under a general sufficient assumption.

Consider the family of sets

$(E)_n^\infty=\{\Omega$ *:open set in* $\mathbb{C}^n$, *with* $0\in\Omega$ *and for any* $z\in\Omega$, *there is a simply connected polydomain* $D_z=D_z^{(1)}\times\ldots\times D_z^{(n)}$ *such that* $\{0,z\}\subset D_z\subset\Omega$

and

$$\vartheta D_z^{(j)} \text{ is } C^\infty - smooth \ (j = 1,2,...,n)\} .$$

It is clear that if Ω is a polydomain of $\mathbb{C}^n$, then $\Omega \in (E)_n^\infty$. For $\Omega \in (E)_n^\infty$, we set

$$\omega_z = \{(x_1,...,x_n) \in \mathbb{C}^n : (x_1,...,x_n, z_1,...,z_n) \in \omega\},$$

and

$$G(\omega,\Omega) = \{z \in \Omega : there\ is\ a\ D_z\ such\ that\ [bD_z]^{-1} \subset \omega_z\}.$$

Here we have used the notation $[bD_z]^{-1} = \{(t_1^{-1},...,t_n^{-1}) : (t_1,...,t_n) \in bD_z\}$, where bD_z denotes the *Shilov boundary* of the polydomain $D_z = D_z^{(1)} \times ... \times D_z^{(n)}$, that is its distinguished boundary $\vartheta\ D_z^{(1)} \times ... \times \vartheta\ D_z^{(n)}$.

Our next *Theorem* can be regarded as a modified form of *Okada-Gawronski-Trautner's Theorem*. Indeed, according to the classical *Theorem* for $n = 1$, *if* Ω *is a domain of* $\mathbb{C}$ ($0 \in \Omega$), *then the* $\mathrm{N}(z)-$*transform of the sequence of partial sums of any* $f \in O\ (\Omega)$, *around* 0, *converges to* f, *compactly on*

$$g(\omega,\Omega) = \{z \in \Omega : \left(\frac{1}{\zeta}, z\right) \in \omega\ ,\ for\ any\ \zeta \in \overline{\mathbb{C}} - \Omega\} .$$

We shall show how to obtain compact convergence for the same as above sequence into the domain

$$G(\omega,\Omega) = \{z \in \Omega : \left(\frac{1}{\zeta}, z\right) \in \omega\ ,\ for\ any\ \zeta \in \vartheta D_z\ and\ some\ D_z\} .$$

Theorem 3.1.15. *If* $\Omega \in (E)_n^\infty$, *then there holds* $G(\omega,\Omega) \subseteq \mathrm{E}_{\mathrm{N}_{\lambda_1},...,\mathrm{N}_{\lambda_\mu}}(O\,(\Omega))$.

In order to prove this *Theorem*, we shall use a *Lemma*, which is a direct consequence of *Cauchy's Integral Formula* on polydomains.

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Lemma 3.1.16. *Let* $\Omega \in (E)_n^{\infty}$ *and let* $w \in \Omega$. *Suppose the functions* $q_{(m_1,\ldots,m_n)}(x,z)$ *are continuous in*

$$[bD_w]^{-1} \times \{w\}$$

$(m_j = 0,1,2,\ldots \text{ and } j = 1,2,\ldots,n)$ *, where* D_w *is a polydomain in* $\mathbb{C}^n$. *If* $f \in O(\Omega)$, *then there holds*

$$\left| f(w) - \sum_{k_1=0}^{m_1} \sigma_{m_1,k_1}^{(1)}(w) \sum_{\nu_1=0}^{k_1} \left(\cdots \left(\sum_{k_n=0}^{m_n} \sigma_{m_n,k_n}^{(n)}(w) \sum_{\nu_n=0}^{k_n} a_{\nu_1,\ldots,\nu_n}^{(f)} w_1^{\nu_1} \ldots w_n^{\nu_n} \right) \cdots \right) \right|$$

$$\le M(f, D_w) \ \sup_{t\in[bD_w]^{-1}} \left| (1-t_1 w_1)^{-1} \ldots (1-t_n w_n)^{-1} - q_{(m_1,\ldots,m_n)}(t_1,\ldots,t_n,z_1,\ldots,z_n) \right|$$

$(m_j = 0,1,2,\ldots \text{ and } j = 1,2,\ldots,n)$, *where* $M(f, D_w)$ *is a constant which depends on* f *and* D_w, *but is independent of* $(m_1,\ldots,m_n)$.

Proof. It suffices to note that

$$f(w) = \frac{1}{(2\pi i)^n} \int_{\zeta=(\zeta_1,\ldots,\zeta_n)\in bD_w} \frac{f(\zeta_1,\ldots,\zeta_n)}{(\zeta_1 - w_1)\ldots(\zeta_n - w_n)} d\zeta_1 \ldots d\zeta_n$$
$$= \frac{1}{(2\pi i)^n} \int_{\zeta=(\zeta_1,\ldots,\zeta_n)\in[bD_w]^{-1}} \frac{f(\zeta_1,\ldots,\zeta_n)}{(1 - w_1\zeta_1)\ldots(1 - w_n\zeta_n)} d\zeta_1 \ldots d\zeta_n$$

and

$$\sum_{k_1=0}^{m_1} \sigma_{m_1,k_1}^{(1)}(w) \sum_{\nu_1=0}^{k_1} \left(\cdots \left(\sum_{k_n=0}^{m_n} \sigma_{m_n,k_n}^{(n)}(w) \sum_{\nu_n=0}^{m_n} a_{\nu_1,\ldots,\nu_n}^{(f)} w_1^{\nu_1} \ldots w_n^{\nu_n} \right) \cdots \right)$$

$$= \frac{1}{(2\pi i)^n} \int_{\zeta=(\zeta_1,\ldots,\zeta_n)\in bD_w} f(\zeta) \left[\sum_{k_1=0}^{m_1} \sigma_{m_1,k_1}^{(1)}(w) \sum_{\nu_1=0}^{k_1} \left(\cdots \left(\sum_{k_n=0}^{m_n} \sigma_{m_n,k_n}^{(n)}(w) \sum_{\nu_n=0}^{k_n} \left(\frac{w_1}{\zeta_1}\right)^{\nu_1} \cdots \left(\frac{w_n}{\zeta_n}\right)^{\nu_n} \right) \cdots \right) \right]$$
$$\zeta_1^{-1} \ldots \zeta_n^{-1} \ d\zeta_1 \ldots d\zeta_n$$

$$= \frac{1}{(2\pi i)^n} \int_{\zeta=(\zeta_1,\ldots,\zeta_n)\in[bD_w]^{-1}} f(\zeta^{-1}) q_{(m_1,\ldots,m_n)}(\zeta, w) \ \zeta_1 \ldots \zeta_n \ \ d\zeta_1 \ldots d\zeta_n \ .$$

Proof of Theorem 3.1.15. Let $f \in O\ (\Omega)$ and let $z^{\circ} = (z_1^{\circ},...,z_n^{\circ}) \in G(\omega,\Omega)$. From the definition of $G(\omega,\Omega)$, it follows that there is a $D_{z^{\circ}}$ with

$$\left[bD_{z^{\circ}}\right]^{-1} \subset \omega_{z^{\circ}}.$$

Consequently, the compact set

$$\left\{(x_1,...,x_n,z_1^{\circ},...,z_n^{\circ}) \in \mathbb{C}^{2n} : (x_1,...,x_n) \in \left[bD_{z^{\circ}}\right]^{-1}\right\}$$

is contained in the open set ω. Applying *Lemma 3.1.16* for $w = z^{\circ}$, we obtain

$$\left| f(z^{\circ}) - \sum_{k_1=0}^{m_1} \sigma_{m_1,k_1}^{(1)}(z^{\circ}) \sum_{\nu_1=0}^{k_1} \left(\cdots \left(\sum_{k_n=0}^{m_n} \sigma_{m_n,k_n}^{(n)}(z^{\circ}) \sum_{\nu_n=0}^{k_n} a_{\nu_1,...,\nu_n}^{(f)} (z_1^{\circ})^{\nu_1} ... (z_n^{\circ})^{\nu_n} \right) \cdots \right) \right|$$

$$\leq M(f,D_{z^{\circ}}) \sup_{t=(t_1,...,t_n)\in\left[bD_{z^{\circ}}\right]^{-1}} \left| (1-t_1 z_1^{\circ})^{-1} ... (1-t_n z_n^{\circ})^{-1} - q_{(m_1,...,m_n)}(t,z^{\circ}) \right|,$$

for $m_j = 0,1,2,...$ and $j = 1,2,...,n$. Since $G(\omega,\Omega)$ is open, there is a compact neighborhood $\overline{U}_{z^{\circ}}$ of z° satisfying

$$\overline{U}_{z^{\circ}} \subset G(\omega,\Omega) \cap D_{z^{\circ}}.$$

Clearly, for any $z \in \overline{U}_{z^{\circ}}$, one can choose D_z equal to $D_{z^{\circ}}$. Repetition of the *Proof* shows that

$$\left| f(z) - \sum_{k_1=0}^{m_1} \sigma_{m_1,k_1}^{(1)}(z) \sum_{\nu_1=0}^{k_1} \left(\cdots \left(\sum_{k_n=0}^{m_n} \sigma_{m_n,k_n}^{(n)}(z) \sum_{\nu_n=0}^{k_n} a_{\nu_1,...,\nu_n}^{(f)} z_1^{\nu_1} ... z_n^{\nu_n} \right) \cdots \right) \right|$$

$$\leq M(f,D_{z^{\circ}}) \sup_{t=(t_1,...,t_n)\in\left[bD_{z^{\circ}}\right]^{-1}} \left| (1-t_1 z_1)^{-1} ... (1-t_n z_n)^{-1} - q_{(m_1,...,m_n)}(t,z) \right|,$$

for $m_j = 0,1,2,...$ and $j = 1,2,...,n$, and for any $z = (z_1,...,z_n) \in \overline{U}_{z^{\circ}}$. Hence,

$$\sup_{z\in\overline{U}_{z^{\circ}}} \left| f(z) - \sum_{k_1=0}^{m_1} \sigma_{m_1,k_1}^{(1)}(z) \sum_{\nu_1=0}^{k_1} \left(\cdots \left(\sum_{k_n=0}^{m_n} \sigma_{m_n,k_n}^{(n)}(z) \sum_{\nu_n=0}^{k_n} a_{\nu_1,...,\nu_n}^{(f)} z_1^{\nu_1} ... z_n^{\nu_n} \right) \cdots \right) \right|$$

$$\leq \sup_{z\in\overline{U}_{z^\circ}} [\, M\left(f, D_{z^\circ}\right) \sup_{t=(t_1,\ldots,t_n)\in\left[bD_{z^\circ}\right]^{-1}} \left| \left(1-t_1 z_1\right)^{-1} \ldots \left(1-t_n z_n\right)^{-1} - q_{(m_1,\ldots,m_n)}(t,z) \right| \,].$$

By passing to the limit, as $m_{\lambda_1} \to \infty, \ldots, m_{\lambda_\mu} \to \infty$, we see that

$$\lim_{m_{\lambda_1}\to\infty,\ldots,m_{\lambda_\mu}\to\infty} \sum_{k_1=0}^{m_1} \sigma^{(1)}_{m_1,k_1}(z) \sum_{\nu_1=0}^{k_1} \left(\cdots \left(\sum_{k_n=0}^{m_n} \sigma^{(n)}_{m_n,k_n}(z) \sum_{\nu_n=0}^{m_n} a^{(f)}_{\nu_1,\ldots,\nu_n} z_1^{\nu_1} \ldots z_n^{\nu_n} \right) \cdots \right) = f(z),$$

uniformly on $\overline{U}_{z^\circ}$, and the *Theorem* follows.

In particular, we have the

Corollary 3.1.17.(a). *If* $\Omega = \Omega_1 \times \ldots \times \Omega_n$ *is a polydomain of* $\mathbb{C}^n$, *with* $0 \in \Omega$, *then*

$$G(\omega, \Omega) \subset \mathrm{E}_{\mathrm{N}_{\lambda_1},\ldots,\mathrm{N}_{\lambda_\mu}} (O(\Omega)).$$

(b). If Ω *is complete Reinhardt domain in* $\mathbb{C}^n$ ($0 \in \Omega$), *then*

$$G(\omega, \Omega) \subset \mathrm{E}_{\mathrm{N}_{\lambda_1},\ldots,\mathrm{N}_{\lambda_\mu}} (O(\Omega)).$$

The final aim of this *Paragraph* is the study of some general and sufficient presuppositions for the global convergence of Padé-type approximants in Runge subdomains of $\mathbb{C}^n$.

Let Ω be a bounded Runge domain in $\mathbb{C}^n$ ($0 \in \Omega$), and $f \in O(\Omega)$. Since Ω is bounded, there are open polydisks

$$\Delta^n(0,d) = \Delta^n\left((0,0,\ldots,0),(d_1,\ldots,d_n)\right),$$

$$\Delta^n(0,\rho) = \Delta^n\left((0,0,\ldots,0),(\rho_1,\ldots,\rho_n)\right) \text{ and}$$

$$\Delta^n(0,r) = \Delta^n((0,0,\ldots,0),(r_1,\ldots,r_n)),$$

such that

$$\Delta^n(0,d) \subset\subset \Delta^n(0,\rho) \subset\subset \Delta^n(0,r) \subset\subset \Omega$$
$$\subset\subset \Delta^n((0,0,\ldots,0),(r_1^{-1},\ldots,r_n^{-1})) \subset\subset \Delta^n((0,0,\ldots,0),(\rho_1^{-1},\ldots,\rho_n^{-1}))$$
$$\subset\subset \Delta^n((0,0,\ldots,0),(d_1^{-1},\ldots,d_n^{-1})).$$

Observe that $f \in O\ (\Delta^n(0,r))$. Following *Theorem 3.1.1*, the distribution T_f is continuous and linear into the Banach space $A(\overline{\Delta^n(0,(\rho_1^{-1},\ldots,\rho_n^{-1}))})$. Since Ω is a Runge domain, there is a sequence

$\{f_k \in O\,(\Delta^n((0,0,\ldots,0),(d_1^{-1},\ldots,d_n^{-1})) : k = 0,1,2,\ldots\}$

such that

$$\lim_{k\to\infty} f_k = f \text{ compactly on } \Omega.$$

Repetition of the *Proof* of *Theorem 3.1.1* (with only formal change to substitute p with f_k) and application of the *Banach-Steinhauss Theorem* imply that

$$\lim_{k\to\infty} T_{f_k} = T_f .$$

Denote by ϖ^{-1} the maximal open neighborhood of

$$\overline{\Delta^n(0,r)} \times \Delta^n((0,0,\ldots,0),(r_1^{-1},\ldots,r_n^{-1}))$$

into which the series

$$\sum_{\nu_1,\ldots,\nu_n=0}^{\infty} x_1^{\nu_1} z_1^{\nu_1} \ldots x_n^{\nu_n} z_n^{\nu_n} \quad (\,|x_j z_j| < 1,\ \text{for}\ j = 1,2,\ldots,n\,)$$

converges compactly to $(1 - x_1 z_1)^{-1} \ldots (1 - x_n z_n)^{-1}$.

Theorem 3.1.18. ([40]) *If* $p_{(m_1,\ldots,m_n)}(x_1,\ldots,x_n,z_1,\ldots,z_n)$ *are analytic polynomials in* $(x_1,\ldots,x_n,z_1,\ldots,z_n)$ *satisfying*

$$\lim_{m_1\to\infty,\ldots,m_n\to\infty} p_{(m_1,\ldots,m_n)}(x_1,\ldots,x_n,z_1,\ldots,z_n) = (1-x_1 z_1)^{-1}\ldots(1-x_n z_n)^{-1}$$

compactly into the open set ω^{-1} *and*

$$\sup_{(m_1,\ldots,m_n)\in\mathrm{N}^n}\left(\sup_{\substack{|s_j|=d_j^{-1}\,(j=1,2,\ldots,n)\\(z_1,\ldots,z_n)\in\Omega}}\left|p_{(m_1,\ldots,m_n)}(s_1,\ldots,s_n,z_1,\ldots,z_n)\right|\right)<\infty$$

for some $d=(d_1,d_2,\ldots,d_n)$ *with* $0<d_j<r_j\quad(j=1,2,\ldots,n)$, *then*

$$\lim_{m_1\to\infty,\ldots,m_n\to\infty} T_f\left(p_{(m_1,\ldots,m_n)}(x_1,\ldots,x_n,z_1,\ldots,z_n)\right)=f(z_1,\ldots,z_n)$$

point-wise in Ω.

Proof. For any $z\in\Omega$, we have

$$\lim_{m_1\to\infty,\ldots,m_n\to\infty} T_f\left(p_{(m_1,\ldots,m_n)}(x_1,\ldots,x_n,z_1,\ldots,z_n)\right)$$

$$=\lim_{m_1\to\infty,\ldots,m_n\to\infty}\ \lim_{k\to\infty} T_{f_k}\left(p_{(m_1,\ldots,m_n)}(x_1,\ldots,x_n,z_1,\ldots,z_n)\right)$$

$$=\lim_{k\to\infty}\ \lim_{m_1\to\infty,\ldots,m_n\to\infty} T_{f_k}\left(p_{(m_1,\ldots,m_n)}(x_1,\ldots,x_n,z_1,\ldots,z_n)\right)$$

$$=\lim_{k\to\infty} f(z_1,\ldots,z_n)\ \text{(by } \mathit{Lemma\ 3.1.6}\text{)}$$

$$=f(z_1,\ldots,z_n).$$

We can now investigate the global point-wise convergence of a sequence of Padé-type approximants to an analytic function defined in a Runge domain:

Theorem 3.1.19. ([40]) *Suppose the generating polynomials*

$$V_{(m_1+1,\dots,m_n+1)}(x_1,\dots,x_n)=\gamma\prod_{j=1}^{n}V_{m_j+1}(x_j)$$

of a Padé-type approximation satisfy

$$(C_n)\qquad \lim_{m_1\to\infty,\dots,m_n\to\infty}\left(1-\frac{V_{m_1+1}(x_1)}{V_{m_1+1}(z_1^{-1})}\right)\dots\left(1-\frac{V_{m_n+1}(x_n)}{V_{m_n+1}(z_n^{-1})}\right)=1$$

compactly in ω^{-1} *and*

$$(B_n)\qquad \sup_{(m_1,\dots,m_n)}\left(\sup_{\substack{|s_j|=d_j^{-1}(j=1,\dots,n)\\(z_1,\dots,z_n)\in\Omega}}\left|1-\frac{V_{m_1+1}(s_1)}{V_{m_1+1}(z_1^{-1})}\right|\dots\left|1-\frac{V_{m_n+1}(s_n)}{V_{m_n+1}(z_n^{-1})}\right|\right)<\infty$$

for some $d=(d_1,\dots,d_n)$ *with* $0<d_j<r_j$ $(1\le j\le n)$. *Then, for any* $f\in O(\Omega)$, *it holds*

$$\lim_{m_1\to\infty,\dots,m_n\to\infty}(m_1,\dots,m_n/m_1+1,\dots,m_n+1)_f(z)=f(z),$$

whenever $z\in\Omega$.

The *Proof* of *Theorem 3.1.19* is an easy consequence of *Corollary 3.1.8* and *Theorem 3.1.18*.

The cases $n=1$ and $n=2$ are of particular significance:

Corollary 3.1.20. (a). *Let* Ω *be a simply connected bounded planar region, containing* 0. *If the generating polynomials* $V_{m+1}(x)$ *of a Padé-type approximation satisfy*

$$(C_1)\qquad \lim_{m\to\infty}\frac{V_{m+1}(x)}{V_{m+1}(z^{-1})}=0$$

compactly in $\omega^{-1}\subset\mathbb{C}^2$ *and*

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(B_1) $$\sup_{m\in\mathbb{N}}\left(\sup_{|s|=d^{-1},z\in\Omega}\left|1-\frac{V_{m+1}(s)}{V_{m+1}(z^{-1})}\right|\right)<\infty$$

for some $d(<r)$, *then for any* $f\in O(\Omega)$ *it holds*

$$\lim_{m\to\infty}(m/m+1)_f(z)=f(z),$$

whenever $z\in\Omega$.

(b). *Let* Ω *be a bounded Runge domain of* $\mathbb{C}^2$, *containing the origin. If the generating polynomials*

$$V_{(m_1+1,m_2+1)}(x_1,x_2)=\gamma\ V_{m_1+1}(x_1)V_{m_n+1}(x_2)$$

of a Padé-type approximation fulfil

$$(C_2)\ \lim_{m_1\to\infty,m_2\to\infty}\frac{V_{m_1+1}(z_1^{-1})V_{m_2+1}(x_2)+V_{m_1+1}(x_1)V_{m_2+1}(z_2^{-1})-V_{m_2+1}(x_1)V_{m_2+1}(x_2)}{V_{(m_1,m_2)}(z_1^{-1},z_2^{-1})}=0$$

compactly in $\omega^{-1}\subset\mathbb{C}^4$ *and*

(B_2) $$\sup_{m_1,m_2\in\mathbb{N}}\left(\sup_{\substack{|s_1|=d_1^{-1},|s_2|=d_2^{-1}\\(z_1,z_2)\in\Omega}}\left|1-\frac{V_{m_1+1}(s_1)}{V_{m_1+1}(z_1^{-1})}\right|\left|1-\frac{V_{m_2+1}(s_2)}{V_{m_2+1}(z_2^{-1})}\right|\right)<\infty,$$

for some (d_1,d_2), *with* $d_1<r_1$ *and* $d_2<r_2$, *then for any* $f\in O(\Omega)$ *there holds*

$$\lim_{m_1\to\infty,m_2\to\infty}(m_1,m_2/m_1+1,m_2+1)_f(z)=f(z),$$

whenever $z\in\Omega$.

3.2. Generalizations: The Analytic Case

3.2.1 The Bergman Kernel Function

Let Ω be any bounded domain in $\mathbb{C}^n$, and let $h(z,\overline{z}) \in C(\overline{\Omega})$ be a positive function in Ω. We consider the Hilbert space $L_h^2(\Omega)$, with inner product

$$\langle f, g \rangle_h = \int_\Omega f\, \overline{g}\, h\, dV ,$$

where dV is the volume element, and the integral is understood as an improper integral. The space $OL_h^2(\Omega)$ of all analytic functions $f \in L_h^2(\Omega)$ is a closed subspace of $L_h^2(\Omega)$ and hence is itself a Hilbert space with finite norm

$$\|f\|_2^{(h)} = \int_\Omega |f|^2 h\, dV.$$

For each $z \in \Omega$, the evaluation map

$$\tau_z : OL_h^2(\Omega) \to \mathbb{C}, \text{ defined by } \tau_z(f) = f(z)$$

is a bounded linear functional on $OL_h^2(\Omega)$. Therefore, by the *Riesz Representation Theorem*, there is a unique element in $OL_h^2(\Omega)$, denoted by $K_\Omega(\cdot, z)$, such that

$$f(z) = \tau_z(f) = \langle f, K_\Omega(\cdot, z) \rangle_h = \int_\Omega f(\zeta)\, \overline{K_\Omega(\zeta, z)}\, h(\zeta, \overline{\zeta})\; dV(\zeta),$$

for all $f \in OL_h^2(\Omega)$. The function

$$K_\Omega : \Omega \times \Omega \to \mathbb{C},$$

with $K_\Omega(\cdot, z) \in OL_h^2(\Omega)$, is called *the Bergman kernel function for* Ω *with respect to the weight* h or simply the *Bergman kernel function*. As it is easily seen, the Bergman kernel function satisfies the following fundamental symmetry property:

$$K_\Omega(\zeta,z)=\overline{K_\Omega(z,\zeta)} \quad \textit{for all } \zeta,z\in\Omega,$$

and hence $K_\Omega(\zeta,z)$ is conjugate analytic in z.

Further, it can be shown in the usual way that there are complete orthonormal systems in $O\,L_h^2(\Omega)$. The Bergman kernel function K_Ω has an interesting representation in terms of such a system:

Theorem 3.2.1. *For any orthonormal basis*

$$\{\varphi_j : j=0,1,2,\ldots\}$$

for $O\,L_h^2(\Omega)$, one has the representation

$$K_\Omega(\zeta,z)=\sum_{j=0}^{\infty}\varphi_j(\zeta)\overline{\varphi_j(z)} \quad (\zeta,z)\in\Omega\times\Omega,$$

with uniform convergence on compact subsets of $\Omega\times\Omega$. For a fixed $z\in\Omega$, this series converges in the L_h^2 − norm with respect to ζ.

Proof. Given any orthonormal basis $\{\varphi_j : j=0,1,2,\ldots\}$ for $O\ L_h^2(\Omega)$, the function K_Ω has the representation

$$K_\Omega(\zeta,z)=\sum_{j=0}^{\infty}\langle K_\Omega(\cdot,z),\varphi_j\rangle_h\,\varphi_j(\zeta) \quad (\zeta,z)\in\Omega\times\Omega.$$

Since

$$\langle K_\Omega(\cdot,z),\varphi_j\rangle_h=\overline{\langle\varphi_j,K_\Omega(\cdot,z)\rangle_h}=\overline{\varphi_j(z)},$$

the representation

$$K_\Omega(\zeta,z)=\sum_{j=0}^{\infty}\varphi_j(\zeta)\overline{\varphi_j(z)} \quad (\zeta,z)\in\Omega\times\Omega$$

follows. For the remaining statement, it is enough to prove uniform boundedness of the partial sums

$$\sum_{j=0}^{m}|\varphi_j(\zeta)|\,|\varphi_j(z)| \qquad (m=0,1,2,...)$$

on $\mathrm{K}\times\mathrm{K}$, for an arbitrary compact $\mathrm{K}\subset\subset\Omega$. Compact convergence then follows by a normality argument.

First, notice that, for all $z\in\Omega$ and all $f\in O\ L_h^2(\Omega)$, the *Cauchy Estimates* imply that

$$|f(z)|\le C_n\,[dist(z,\vartheta\Omega)]^{-n}\,\|f\|_2^{(h)},$$

where the constant C_n depends only on the dimension n. Since

$$\|K_\Omega(\cdot,z)\|_2^{(h)}=\|\tau\|_2^{(h)}=\sup\{|f(z)|: f\in O\ L_h^2(\Omega),\|f\|_2^{(h)}\le 1\},$$

we infer that the Bergman kernel K_Ω satisfies the estimate

$$\|K_\Omega(\cdot,z)\|_2^{(h)}\le C_n[dist(z,\vartheta\Omega)]^{-n} \qquad (z\in\Omega).$$

Let now K be a compact subset of Ω. Then $dist(\mathrm{K},\vartheta\Omega)>0$, and, from the above estimate, it follows that there is a constant σ_{K} such that

$$\|K_\Omega(\cdot,z)\|_2^{(h)}\le\sigma_{\mathrm{K}} \qquad (z\in\mathrm{K}).$$

Since

$$\left[\|K_\Omega(\cdot,z)\|_2^{(h)}\right]^2=\sum_{j=0}^{\infty}|\langle K_\Omega(\cdot,z),\varphi_j\rangle|^2=\sum_{j=0}^{\infty}\left|\overline{\langle\varphi_j,K_\Omega(\cdot,z)\rangle}\right|^2=\sum_{j=0}^{\infty}|\varphi_j(z)|^2,$$

we obtain

$$\sup_{z\in\Omega}\sum_{j=0}^{\infty}|\varphi_j(z)|^2\le\sigma_{\mathrm{K}}\ .$$

So, by the *Cauchy-Schwarz Inequality*, one has

$$\sum_{j=0}^{\infty}\left|\varphi_j(z)\right|\left|\varphi_j(\zeta)\right| \le \left(\sum_{j=0}^{\infty}\left|\varphi_j(z)\right|^2\right)^{1/2}\left(\sum_{j=0}^{\infty}\left|\varphi_j(\zeta)\right|^2\right)^{1/2} \le \sigma_{\mathrm{K}}{}^2,$$

for all $z,\zeta \in \mathrm{K}$, which completes the *Proof* of the *Theorem*.

Let us now give some examples of computation for the Bergman kernel function, confining ourselves to complete circular domains Ω and to the weight function $h \equiv 1$. Recall that a domain $\Omega \subset \mathbf{C}^n$ is said to be *circular* if $e^{i\theta} z \in \Omega$ whenever $z \in \Omega$ and θ is real. It is convenient to take the set of monomials

$$\left\{ A_k z^k = A_{(k_1,\dots,k_n)} z_1^{k_1} \dots z_n^{k_n} : k = (k_1,\dots,k_n),\, k_\nu = 0,1,2,\dots (\nu = 1,2,\dots,n) \right\}$$

as a complete orthonormal system, with

$$A_k = \left(\int_\Omega \left|z^k\right|^2 dV(z)\right)^{-\frac{1}{2}}.$$

The completeness of this system follows from the fact that analytic functions can be expanded in a Taylor series for the class of domains under consideration. The orthonormality follows from A_k's definition and from the fact that

$$\int_\Omega z^k \bar{z}^{k'} dV(z) = 0, \quad \text{for} \quad k \ne k'.$$

Theorem 3.2.2. *The Bergman kernel function for the polydisk* $\Delta^n(0,r), r = (r_1,\dots,r_n)$, *is*

$$K_{\Delta^n(0,r)}(\zeta,z)=\frac{1}{\pi^n}\prod_{\nu=1}^{n}\frac{r_\nu^{\,2}}{\left(r_\nu^{\,2}-\zeta_\nu\overline{z_\nu}\right)^2}\qquad \left(\zeta,z\in\Delta^n(0,r)\right).$$

Proof. For the polydisk $\Delta^n(0,r)$, we get from the definition of A_k that

$$A_k^{\,2}=\frac{1}{\pi^n}\prod_{\nu=1}^{n}\frac{k_\nu+1}{r_\nu^{2k_\nu+2}}.$$

Then, by *Theorem 3.2.1* (here $x_\nu=\zeta_\nu\overline{z_\nu}\,r_\nu^{-2}$),

$$K_{\Delta^n(0,r)}(\zeta,z)=\sum_k\frac{1}{\pi^n}\prod_{\nu=1}^{n}\frac{k_\nu+1}{r_\nu^{2k_\nu+2}}\zeta^k\overline{z}^k$$

$$=\frac{1}{\pi^n r_1^2\ldots r_n^2}\sum_k\frac{\vartheta^n}{\vartheta x_1\ldots\vartheta x_n}x^k$$

$$=\frac{1}{\pi^n r_1^2\ldots r_n^2}\frac{\vartheta^n}{\vartheta x_1\ldots\vartheta x_n}\left(\frac{1}{(1-x_1)\ldots(1-x_n)}\right)$$

$$=\frac{1}{\pi^n r_1^2\ldots r_n^2}\frac{1}{(1-x_1)^2\ldots(1-x_n)^2}$$

$$=\frac{1}{\pi^n}\prod_{\nu=1}^{n}\frac{r_\nu^2}{\left(r_\nu^2-\zeta_\nu\overline{z_\nu}\right)^2},$$

for all $(\zeta,z)\in\Delta^n(0,r)\times\Delta^n(0,r)$.

Theorem 3.2.3. *The Bergman kernel function for the ball* $B^n(0,R)$, $R>0$, *is*

$$K_{B^n(0,R)}=\frac{n!\,R^2}{\pi^n\left(R^2-\langle\zeta,z\rangle\right)^{n+1}}=\frac{n!\,R^2}{\pi^n\left(R^2-\zeta_1\overline{z_1}-\ldots\zeta_n\overline{z_n}\right)^{n+1}}$$

for any $\zeta=(\zeta_1,\ldots,\zeta_n)\in B^n(0,R)$ *and* $z=(z_1,\ldots,z_n)\in B^n(0,R)$, *where we have used the* *notation* $\langle \zeta, z\rangle$ *for the usual inner product of* $\mathbb{C}^n$:

$$\langle \zeta, z\rangle := \sum_{\nu=1}^{n} \zeta_\nu \overline{z_\nu}.$$

Proof. For the ball $B^n(0,R)$, the definition of A_k gives us that

$$A_k^2 = \frac{(k_1+\ldots+k_n+n)!}{k_1!\ldots k_n!\pi^n R^{2(k_1+\ldots+k_n+n)}}.$$

So, by *Theorem 3.2.1* (here $x = R^{-2}\sum_{\nu=1}^{n}\zeta_\nu\overline{z_\nu} = R^{-2}\langle\zeta,z\rangle$, $|x|<1$),

$$\begin{aligned}
K_{B^n(0,R)}(\zeta,z) &= \frac{1}{\pi^n R^{2n}}\sum_{k}\frac{(k_1+\ldots+k_n+n)!}{k_1!\ldots k_n! R^{2(k_1+\ldots+k_n)}}\zeta^k\overline{z}^k \\
&= \frac{1}{\pi^n R^n}\sum_{m=0}^{\infty}\frac{(m+1)\ldots(m+n)}{R^{2n}}\sum_{k_1+\ldots+k_n=m}\frac{m!}{k_1!\ldots k_n!}\zeta^k\overline{z}^k \\
&= \frac{1}{\pi^n R^n}\sum_{m=0}^{\infty}\frac{d^n}{dx^n}\left(x^m\right) \\
&= \frac{1}{\pi^n R^n}\frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) \\
&= \frac{n!}{\pi^n R^{2n}(1-x)^{n+1}} \\
&= \frac{n!\,R^2}{\pi^n\left(R^2-\langle\zeta,z\rangle\right)^{n+1}} \\
&= \frac{n!\,R^2}{\pi^n\left(R^2-\zeta_1\overline{z_1}-\ldots-\zeta_n\overline{z_n}\right)^{n+1}}
\end{aligned}$$

for all $(\zeta,z)\in B^n(0,R)\times B^n(0,R)$, which completes the *Proof* of the *Theorem*.

3.2.2. Haar's condition

Up to this point we have been considering the approximation of functions by ordinary analytic polynomials. In the one complex variable case, analytic polynomials of degree $\leq m$ are of course simply linear combinations of the functions $1, x, x^2, ..., x^m$. It is natural to generalize the concept of a polynomial to include linear combinations of other prescribed complex functions, say

$$g_0, g_1, g_2, ..., g_m.$$

We shall always assume that such functions are continuous on some fixed compact metric space S, containing at least $(m+1)$ points. Their linear combinations

$$\sum_{i=0}^{m} c_i\, g_i$$

will be termed *generalized polynomials over* S $(c_i \in S)$.

Definition 3.2.4. *Let* X_{m+1} *be the* $(m+1)-$*dimensional complex vector space of generalized polynomials over* S, *which is generated by the continuous functions* $g_0, g_1, g_2, ..., g_m$. *We say that* X_{m+1} *satisfies the Haar condition over* S, *if every function in* X_{m+1} *has at most* m *roots in* S; *the functional discret set*

$$\{g_0, g_1, g_2, ..., g_m\}$$

is sometimes termed a Tchebycheff system.

A systematic investigation of Haar's condition is incorporated in [31] and [92]. His deeper advantage is connected with the characterization of best approximations. In order to limit the *Section* size, this important topic $-$ for which fortunately excellent references are available $-$ had to be omitted. Here, we will only discuss the "interpolatory" interpretation of this important condition.

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Theorem 3.2.5. *The following are equivalent:*

(a). X_{m+1} *satisfies the Haar condition over* S.

(b). $\{g_0, g_1, \ldots, g_m\}$ *is a Tchebycheff system.*

(c). *If* $\{s_0, s_1, \ldots, s_m\}$ *is any finite collection of pair-wise distinct points in* S, *then*

$$\det\left[g_j(s_k)\right]_{k,j} = \begin{vmatrix} g_o(s_o) & g_1(s_o) & \ldots & g_m(s_o) \\ g_o(s_1) & g_1(s_1) & \ldots & g_m(s_1) \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ g_o(s_m) & g_1(s_m) & \ldots & g_m(s_m) \end{vmatrix} \neq 0.$$

(d). *If* $\{s_0, s_1, \ldots, s_m\}$ *is any finite collection of pair-wise distinct points in* S *and if* $y_0, y_1, \ldots, y_m$ *are arbitrary complex numbers, then there exists only one generalized polynomial*

$$g = \sum_{i=0}^{m} c_i\, g_i \ \ in \ \ X_{m+1}$$

such that $g(s_k) = y_k \quad for \ \ k = 0,1,2,\ldots,m$.

Proof. It is clear that *(a)* $\Leftrightarrow$ *(b)*. To complete the *Proof*, we shall show that *(d)* $\Leftrightarrow$ *(c)* and *(a)* $\Leftrightarrow$ *(c)*.

First, assume *(d)* We see that the interpolation condition $g(s_k) = y_k \quad (for \ \ k = 0,1,2,\ldots,m)$ is equivalent to the existence of a unique solution for the linear system

$$\sum_{j=0}^{m} c_j g_j(s_k) = y_k \qquad (0 \leq k \leq m).$$

This is equivalent to the non-vanishing of the determinant

$$\det\left[g_j(s_k) = y_k\right]_{k,j}.$$

Hence, *(d)* $\Leftrightarrow$ *(c)*. Next, a necessary and sufficient condition for *(a)* is that any generalized polynomial

$$g = \sum_{i=0}^{m} c_i \, g_i \in X_{m+1},$$

with $(m+1)$ distinct roots

$$s_0, s_1, \ldots, s_m,$$

is identically equal to the zero function. In other words, a necessary and sufficient condition for *(a)* is that the homogeneous linear system (with respect to c_j's)

$$\sum_{j=0}^{m} c_j g_j(s_k) = y_k \qquad (k = 0,1,2,\ldots,m),$$

has the unique solution $(c_0, c_1, \ldots, c_m) = (0,0,\ldots,0)$. But, this is equivalent to

$$\det\lfloor g_j(s_k) = y_k \rfloor_{k,j} \neq 0,$$

that is *(a)* $\Leftrightarrow$ *(c)*, which ends the *Proof* of the *Theorem*.

It is important to know if Haar's condition is satisfied over an arbitrary compact subset S of $\mathbf{C}^n$. The following result, due to Mairhuber and to Sieklucki, shows that only a very small class of compact sets in $\mathbf{C}^n$ can be considered:

Theorem 3.2.6. ([92]) *Let S be a compact subset of $\mathbf{C}^n$. A necessary and sufficient condition for the existence of a $(m+1)-$dimensional complex vector space X_{m+1} satisfying Haar's condition over S is the homeomorphic identity of S with a closed subset of the unit planar circle.*

3.2.3. Generalized Padé-type Approximants to Analytic L^2-Functions

From the point of view of integral representations, a major difference between the case of one complex variable and the general case is due to the fact that in one variable there is essentially only one kernel function $-$the Cauchy kernel $(1-xz)^{-1}$ $-$, while in several variables one has great freedom to modify, by a basically algebraic procedure, the original potential theoretic kernels.

In particular, for complex dimension one, Padé and Padé-type approximation theory is based on the choice of polynomials interpolating the Cauchy kernel function in x . For dimension higher than one, we were faced with the problem to develop a coordinate Padé or Padé-type approximation method in polydisks, by interpolating the Cauchy kernel function

$(1-x_1z_1)^{-1}...(1-x_nz_n)^{-1}$.

But, *Section 3.1* showed that approximation results, obtained by this method, lead to extremely complicated computations if $n \geq 3$. So, in the following, we will suggest a totally different method that applies to all bounded domains Ω (and not only on polydisks) of $\mathbb{C}^n$ and can be considered as a natural extension of Brezinski' s ideas from one to several variables.

The general idea is to replace the Cauchy kernel with the Bergman kernel function

$K_\Omega(z,x)$.

Even though the details of this method involve a general theoretical machinery, it should be stressed that its applications to the approximation of functions are readily accessible. But, on the other hand, the treatment of this method while having certain advantages is limited to the special class of analytic functions on Ω which are in $L^2(\Omega)$. Further, the computation of the Bergman kernel function for arbitrary domains is a difficult problem. However, at least for the case of bounded circular complete domains in $\mathbb{C}^n$, we are able to obtain concrete results.

Let Ω be a bounded domain in $\mathbb{C}^n$. As in *Paragraph 3.2.1*, we will denote by $OL^2(\Omega)$ the collection of all functions f analytic in Ω with finite norm:

$$\|f\|_2 := \left(\int_\Omega |f|^2 \, dV \right)^{1/2},$$

where dV is the volume element; $OL^2(\Omega)$ is a Hilbert space with inner product

$$\langle f, g \rangle := \int_\Omega f \, \overline{g} \, dV \qquad \left(f, g \in OL^2(\Omega) \right).$$

Let $f \in OL^2(\Omega)$ and suppose

$$\{\varphi_j(z) : j = 0,1,2,\ldots\}$$

is a complete orthonormal system in $OL^2(\Omega)$. If $a_j^{(f)}$ are the Fourier coefficients of f with respect to this system, then the series

$$f(z) = \sum_{j=0}^{\infty} a_j^{(f)} \varphi_j(z) \qquad (z \in \Omega)$$

converges in the norm $\|\cdot\|_2$. Let us introduce the linear functional

$$T_f : \overline{\Phi(\mathbb{C}^n)} \to \mathbb{C}^n : \overline{\varphi_j(z)} \mapsto T_f\left(\overline{\varphi_j(z)}\right) := a_j^{(f)},$$

where $\overline{\Phi(\mathbb{C}^n)}$ is the complex vector space generated by all finite complex combinations of $\overline{\varphi_j}$'s. If

$$\overline{p(x)} = \sum_{\nu=0}^{m} \beta_\nu \overline{\varphi_j(x)} \in \overline{\Phi(\mathbb{C}^n)},$$

then

$$\left| T_f\left(\overline{p(x)}\right) \right| = \left| T_f\left(\sum_{\nu=0}^{m} \beta_\nu \overline{\varphi_j(x)} \right) \right| = \left| \sum_{\nu=0}^{m} \beta_\nu \, T_f\left(\overline{\varphi_\nu(x)}\right) \right|$$

$$= \left| \sum_{\nu=0}^{m} \beta_\nu a_\nu^{(f)} \right| = \left| \sum_{\nu=0}^{m} \beta_\nu \int_\Omega f\, \overline{\varphi_\nu}\, dV \right|$$

$$= \left| \int_\Omega f \left(\sum_{\nu=0}^{m} \beta_\nu \overline{\varphi_\nu} \right) dV \right| = \left| \int_\Omega f\, \overline{p}\, dV \right|.$$

It follows, from *Hölder's Inequality*, that

$$\left| T_f \left(\overline{p(x)} \right) \right| \leq \left[\int_\Omega |f|^2 dV \right]^{1/2} \left[\int_\Omega |\overline{p}|^2 dV \right]^{1/2} = \|f\|_2 \|\overline{p}\|_2$$

and therefore, by the *Hahn-Banach Theorem*, T_f extends to a linear continuous functional on the Hilbert space $L^2(\Omega)$ of all complex-valued functions g in Ω, with inner product

$$\langle g, h \rangle := \int_\Omega g\, \overline{h}\, dV \quad \left(g, h \in L^2(\Omega) \right).$$

Obviously, for any $z \in \Omega$ fixed, the Bergman kernel function $K_\Omega(z, x)$ belongs to $L^2(\Omega)$ and thus, one can define the number

$$T_f \left(K_\Omega(z, x) \right),$$

where the extended functional T_f acts on the variable x. Furthermore, by continuity, there holds

$$f(z) = \sum_{j=0}^{\infty} a_j^{(f)} \varphi_j(z) = \sum_{j=0}^{\infty} T_f \left(\overline{\varphi_j(x)} \right) \varphi_j(z) = T_f \left(\sum_{j=0}^{\infty} \overline{\varphi_j(x)} \varphi_j(z) \right)$$

and, by *Theorem 3.2.1*,

$$f(z) = T_f \left(K_\Omega(z, x) \right).$$

Thus, computing $f(z)$ for a fixed value of z is nothing else than computing $T_f \left(K_\Omega(z, x) \right)$.

It arises in practice that only a few Fourier coefficients $a_j^{(f)}$ of $f \in OL^2(\Omega)$ are known or that the Fourier series expansion of f (with respect to the basis $\{\varphi_j(z) : j = 0,1,2,...\}$) converges too slowly.

Thus, the function $K_\Omega(z,x)$ has to be replaced by a simpler expression. To do so, for any $m=0,1,2,\ldots$, consider the $(m+1)-$dimensional complex vector space $\overline{\Phi}_{m+1}$, generated by the Tchebycheff system

$$\left\{\overline{\varphi_0},\overline{\varphi_1},\ldots,\overline{\varphi_m}\right\}.$$

Denoting by Z_{m+1} the analytic set

$$\bigcup_{0\le j\le m} Ker\overline{\varphi_j}$$

(: $Ker\overline{\varphi_j}$ is the kernel of $\overline{\varphi_j}$), suppose that $\overline{\Phi}_{m+1}$ satisfies the Haar condition into a finite set of pair-wise distinct points

$$\mathrm{M}_{m+1}=\left\{\pi_{m,0},\pi_{m,1},\ldots,\pi_{m,m}\right\}\subset\Omega.$$

This means that every function in $\overline{\Phi}_{m+1}$ has at most m roots in M_{m+1}. One can, for example, choose the set M_{m+1} so that

$$\mathrm{M}_{m+1}\cap Z_{m+1}=\varnothing.$$

It follows that, for any $z\in\Omega$, there is a unique

$$g_m(x,z)=\sum_{j=0}^{m}c_j^{(m)}(z)\overline{\varphi_j(x)}\in\ \overline{\Phi}_{m+1},$$

such that

$$g_m\left(\pi_{m,k},z\right)=K_\Omega\left(z,\pi_{m,k}\right)\qquad\text{for any } k\le m,$$

or explicitly

$$\sum_{j=0}^{m}c_j^{(m)}(z)\overline{\varphi_j\left(\pi_{m,k}\right)}=K_\Omega\left(z,\pi_{m,k}\right)\qquad(\text{for } k=0,1,\ldots,m).$$

Note that to find $c_j^{(m)}(z)$ it is enough to solve the above linear system: a necessary and sufficient condition for the existence of a unique solution is that the determinant

$$\det\left[\varphi_j\left(\pi_{m,k}\right)\right]_{k,j}=\begin{vmatrix}\overline{\varphi_0\left(\pi_{m,0}\right)} & \overline{\varphi_1\left(\pi_{m,0}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,0}\right)}\\ \overline{\varphi_0\left(\pi_{m,1}\right)} & \overline{\varphi_1\left(\pi_{m,1}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,1}\right)}\\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\\ \overline{\varphi_0\left(\pi_{m,m}\right)} & \overline{\varphi_1\left(\pi_{m,m}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,m}\right)}\end{vmatrix}$$

is different from zero. This condition is equivalent to the Haar condition for $\overline{\Phi}_{m+1}$ into the set M_{m+1}. Obviously, for any $z\in\Omega$, there holds

$$c_j^{(m)}(z)$$

$$=\frac{\begin{vmatrix}\overline{\varphi_0\left(\pi_{m,0}\right)} & \overline{\varphi_1\left(\pi_{m,0}\right)} & \cdots & \overline{\varphi_{j-1}\left(\pi_{m,0}\right)} & K_\Omega\left(z,\pi_{m,0}\right) & \overline{\varphi_{j+1}\left(\pi_{m,0}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,0}\right)}\\ \overline{\varphi_0\left(\pi_{m,1}\right)} & \overline{\varphi_1\left(\pi_{m,1}\right)} & \cdots & \overline{\varphi_{j-1}\left(\pi_{m,1}\right)} & K_\Omega\left(z,\pi_{m,1}\right) & \overline{\varphi_{j+1}\left(\pi_{m,1}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,1}\right)}\\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots\\ \overline{\varphi_0\left(\pi_{m,m}\right)} & \overline{\varphi_1\left(\pi_{m,m}\right)} & \cdots & \overline{\varphi_{j-1}\left(\pi_{m,m}\right)} & K_\Omega\left(z,\pi_{m,m}\right) & \overline{\varphi_{j+1}\left(\pi_{m,m}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,m}\right)}\end{vmatrix}}{\begin{vmatrix}\overline{\varphi_0\left(\pi_{m,0}\right)} & \overline{\varphi_1\left(\pi_{m,0}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,0}\right)}\\ \overline{\varphi_0\left(\pi_{m,1}\right)} & \overline{\varphi_1\left(\pi_{m,1}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,1}\right)}\\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots\\ \overline{\varphi_0\left(\pi_{m,m}\right)} & \overline{\varphi_1\left(\pi_{m,m}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,m}\right)}\end{vmatrix}}$$

or alternatively,

$$\left(\mathrm{E}_j\right)\qquad c_j^{(m)}(z)=\frac{K_\Omega\left(z,\pi_{m,0}\right)}{\overline{\varphi_j\left(\pi_{m,0}\right)}}+\frac{K_\Omega\left(z,\pi_{m,1}\right)}{\overline{\varphi_j\left(\pi_{m,1}\right)}}+\cdots+\frac{K_\Omega\left(z,\pi_{m,m}\right)}{\overline{\varphi_j\left(\pi_{m,m}\right)}}=\sum_{k=0}^{m}\frac{K_\Omega\left(z,\pi_{m,k}\right)}{\overline{\varphi_j\left(\pi_{m,k}\right)}}$$

$$\left(j=0,1,\ldots,n\right).$$

In the formula

$$f(z)=T_f\left(K_\Omega(z,x)\right)$$

let us now replace $K_\Omega(z,x)$ by $g_m(x,z)$ to obtain

$$T_f\left(g_m(x,z)\right) = T_f\left(\sum_{j=0}^{m} c_j^{(m)}\overline{\varphi_j(x)}\right) = \sum_{j=0}^{m} a_j^{(f)} c_j^{(m)}(z).$$

It is easily seen that each $c_j^{(m)}(z)$ belongs to $OL^2(\Omega)$ and therefore, for each j, there are Fourier constants $s_\nu^{(j,m)}$ such that

$$c_j^{(m)}(z) = \sum_{\nu=0}^{\infty} s_\nu^{(j,m)} \varphi_\nu(z), \qquad z \in \Omega.$$

It follows that

$$T_f\left(g_m(x,z)\right) = \sum_{j=0}^{m} a_j^{(f)} \sum_{\nu=0}^{\infty} s_\nu^{(j,m)} \varphi_\nu(z) = \sum_{\nu=0}^{\infty}\left(\sum_{j=0}^{m} s_\nu^{(j,m)} a_j^{(f)}\right)\varphi_\nu(z) \qquad (z \in \Omega).$$

We may now give the following

Definition 3.2.7. *Any function* $(GPTA/m)_f(z)$, *defined by*

$$(GPTA/m)_f(z) = T_f\left(g_m(x,z)\right) = \sum_{j=0}^{m} a_j^{(f)} c_j^{(m)}(z),$$

is called a generalized Padé-type approximant to $f \in OL^2(\Omega)$, *with generating system*

$$\mathrm{M}_{m+1} = \{\pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}\}.$$

If

$$\sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} \langle f, \varphi_j \rangle \sum_{k=0}^{m} \frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}} = 0 \qquad \textit{for every } \nu = 0,1,\ldots,m,$$

then $T_f\left(g_m(x,z)\right)$ *is said to be a Padé-type approximant to* f, *with generating system* M_{m+1} *and is noted by*

$$(PTA/m)_f(z).$$

According to our preceding discussion, we have the following.

Theorem 3.2.8.(a). *A generalized Padé-type approximant* $(GPTA/m)_f(z)$ *to*

$$f(z)=\sum_{\nu=0}^{\infty}a_{\nu}^{(f)}\varphi_{\nu}(z)\in OL^2(\Omega), \quad \varphi_{\nu}(z)\in OL^2(\Omega),$$

is an analytic function in $L^2(\Omega)$.

(b). *Any Padé-type approximant* $(PTA/m)_f(z)$ *to* $f\in OL^2(\Omega)$ *is a generalized Padé-type approximant.*

(c). *If*

$$\sum_{\nu=0}^{\infty}\beta_{\nu}^{(m,f)}\varphi_{\nu}(z)$$

is the Fourier expansion of a Padé-type approximant $(PTA/m)_f(z)$ *to*

$$f(z)=\sum_{\nu=0}^{\infty}a_{\nu}^{(f)}\varphi_{\nu}(z)\in OL^2(\Omega),$$

with respect to the orthonormal basis $\{\varphi_{\nu}(z):\nu=0,1,2,...\}$, *then there holds*

$$\beta_{\nu}^{(m,f)}=a_{\nu}^{(f)}, \quad \text{for any } \nu=0,1,2,...,m.$$

Proof. It is clear that any Padé-type approximant to $f\in OL^2(\Omega)$ is a generalized Padé-type approximant, and any generalized Padé-type approximant to an $f\in OL^2(\Omega)$ belongs to $OL^2(\Omega)$.Now observe that, for any $\nu\geq 0$, we have

$$a_{\nu}^{(f)}=\int_{\Omega}f\,\overline{\varphi_{\nu}}\,dV=\langle f,\varphi_{\nu}\rangle$$

and

$$\beta_\nu^{(m,f)} = \sum_{j=0}^{m} s_\nu^{(m,j)} a_j^{(f)} = \sum_{j=0}^{m}\left(\int_\Omega c_j^{(m)}(z)\,\overline{\varphi_\nu(z)}\,dV(z)\right)\left(\int_\Omega f(w)\,\overline{\varphi_j(w)}\,dV(w)\right)$$

$$= \sum_{j=0}^{m}\left(\int_\Omega \sum_{k=0}^{m} \frac{K_\Omega(z,\pi_{m,k})}{\varphi_j(\pi_{m,k})}\,\overline{\varphi_\nu(z)}\,dV(z)\right)\left(\int_\Omega f(w)\overline{\varphi_j(w)}dV(w)\right)$$

$$= \sum_{j=0}^{m}\sum_{k=0}^{m} \frac{1}{\varphi_j(\pi_{m,k})}\langle K_\Omega(\cdot,\pi_{m,k}),\varphi_\nu\rangle\,\langle f,\varphi_j\rangle\,.$$

Since

$$\langle K_\Omega(\cdot,\pi_{m,k}),\varphi_\nu(\cdot)\rangle = \overline{\varphi_\nu(\pi_{m,k})},$$

we see that, for any $\nu = 0,1,2,\ldots,m$, it holds

$$\beta_\nu^{(m,f)} = \sum_{j=0}^{m}\sum_{k=0}^{m} \frac{1}{\varphi_j(\pi_{m,k})}\overline{\varphi_\nu(\pi_{m,k})}\langle f,\varphi_j\rangle$$

$$= a_\nu^{(f)} + \sum_{\substack{j=0\\(j\neq\nu)}}^{m}\langle f,\varphi_j\rangle \sum_{k=0}^{m}\frac{\overline{\varphi_\nu(\pi_{m,k})}}{\varphi_j(\pi_{m,k})} = a_\nu^{(f)}$$

which ends the *Proof*.

This *Theorem* justifies the notation "Padé-type approximant" to $f \in OL^2(\Omega)$.

Notice that to compute a generalized Padé-type approximant $(GPTA/m)_f(z)$ to $f(z)$ it suffices to know only the Fourier coefficients

$$a_0^{(f)}, a_1^{(f)}, \ldots, a_m^{(f)}$$

of f and the functions

$$c_0^{(m)}(z), c_1^{(m)}(z), \ldots, c_m^{(m)}(z),$$

resulting from the solution of the equations $(E_0),(E_1),\ldots,(E_m)$, respectively.

Evidently, the free choice of the orthonormal basis

$$\{\varphi_j(z) : j = 0,1,2,...\}$$

and the generating system

$$S_{m+1} = \{\pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}\}$$

may lead to a better approximation of the function. The best choice is a general and difficult question which is not studied herein. However, *Theorem 3.2.9* gives a first theoretical account concerning the error of the approximation. Another problem connected with the choice of the basis and the generating system is the convergence of generalized Padé-type approximants. Some attempts to solve this problem will be presented below in *Theorem 3.2.11* and its *Corollaries 3.2.12* and *3.2.13*.

Theorem 3.2.9.(a). *The error of a generalized Padé-type approximation is*

$$T_f(g_m(x,z)) - f(z) = \sum_{\nu=0}^{\infty}\left[\sum_{j=0}^{m}\langle f,\varphi_j\rangle\sum_{k=0}^{m}\frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}} - \langle f,\varphi_\nu\rangle\right]\varphi_\nu(z).$$

(b). *The error of a Padé-type approximation equals*

$$T_f(g_m(x,z)) - f(z) = \sum_{\nu=m+1}^{\infty}\left[\sum_{j=0}^{m}\langle f,\varphi_j\rangle\sum_{k=0}^{m}\frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}} - \langle f,\varphi_\nu\rangle\right]\varphi_\nu(z).$$

Proof. Let $f \in OL^2(\Omega)$.

*(a)***.** If

$$(GPTA/m)_f(z)$$

is a generalized Padé-type approximant to $f(z)$, then

$$(GPTA/m)_f(z) - f(z) = T_f(g_m(x,z)) - f(z)$$

$$= \sum_{\nu=0}^{\infty} \beta_\nu^{(m,f)} \varphi_\nu(z) - \sum_{\nu=0}^{\infty} a_\nu^{(f)} \varphi_\nu(z)$$

$$= \sum_{\nu=0}^{\infty} \left[\sum_{j=0}^{m} s_\nu^{(m,j)} a_j^{(f)} - a_\nu^{(f)} \right] \varphi_\nu(z)$$

$$= \sum_{\nu=0}^{\infty} \left[\sum_{j=0}^{m} \left(\int_\Omega c_j^{(m)}(w) \overline{\varphi_\nu(w)}\, dV(w) \right) \left(\int_\Omega f(\zeta) \overline{\varphi_j(\zeta)}\, dV(\zeta) \right) - \left(\int_\Omega f(\xi) \overline{\varphi_\nu(\xi)}\, dV(\xi) \right) \right] \varphi_\nu(z)$$

$$= \sum_{\nu=0}^{\infty} \left[\sum_{j=0}^{m} \sum_{k=0}^{m} \left(\int_\Omega \frac{K_\Omega(w, \pi_{m,k}) \overline{\varphi_\nu(w)}}{\varphi_j(\pi_{m,k})}\, dV(w) \right) \left(\int_\Omega f(\zeta) \overline{\varphi_j(\zeta)}\, dV(\zeta) \right) - \left(\int_\Omega f(\xi) \overline{\varphi_\nu(\xi)}\, dV(\xi) \right) \right] \varphi_\nu(z)$$

$$= \sum_{\nu=0}^{\infty} \left[\sum_{j=0}^{m} \sum_{k=0}^{m} \frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}} \langle f, \varphi_j \rangle - \langle f, \varphi_\nu \rangle \right] \varphi_\nu(z)$$

$$= \sum_{\nu=0}^{\infty} \left[\sum_{j=0}^{m} \langle f, \varphi_j \rangle \sum_{k=0}^{m} \frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}} - \langle f, \varphi_\nu \rangle \right] \varphi_\nu(z).$$

(β). If

$$(PTA/m)_f(z)$$

is a Padé-type approximant to $f(z)$, then, by *Theorem 3.2.8.(c)*, we have

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$$\begin{aligned}(PTA/m)_f(z)-f(z) &= T_f\left(g_m(x,z)\right)-f(z)\\ &= \sum_{\nu=0}^{\infty}\beta_\nu^{(m,f)}\varphi_\nu(z)-\sum_{\nu=0}^{\infty}a_\nu^{(f)}\varphi_\nu(z)\\ &= \sum_{\nu=m+1}^{\infty}\left[\sum_{j=0}^{\infty}s_\nu^{(m,j)}a_j^{(f)}-a_\nu^{(f)}\right]\varphi_\nu(z).\end{aligned}$$

Repetition of the *Proof* of *(a)* gives

$$(PTA/m)_f(z)-f(z)=\sum_{\nu=m+}^{\infty}\left[\sum_{j=0}^{m}\langle f,\varphi_j\rangle\sum_{k=0}^{m}\frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}}-\langle f,\varphi_\nu\rangle\right]\varphi_\nu(z).$$

We can now immediately obtain bounds for the errors:

Corollary 3.2.10. *Let* $\mathrm{K}\subset\subset\Omega$ *and let* $f\in OL^2(\Omega)$.

(a). If $(GPTA/m)_f(z)$ *is a generalized Padé-type approximant* to $f(z)$, *then there is a positive constant* $\sigma(\mathrm{K})$, *depending only on the compact set* K, *such that*

$$\sup_{z\in\mathrm{K}}\left|(GPTA/m)_f(z)-f(z)\right|\le\sigma(\mathrm{K})\|f\|_2\left\{\sum_{\nu=0}^{\infty}\left\|\sum_{j=0}^{m}\varphi_j\left(\sum_{k=0}^{m}\frac{\varphi_\nu(\pi_{m,k})}{\varphi_j(\pi_{m,k})}\right)-\varphi_\nu\right\|_2^2\right\}^{\frac{1}{2}}.$$

(b). If $(PTA/m)_f(z)$ *is a Padé-type approximant to* $f(z)$, *then there is a positive constant* $\tau(\mathrm{K})$, *depending only on the compact set* K, *such that*

$$\sup_{z\in\mathrm{K}}\left|(PTA/m)_f(z)-f(z)\right|\le\tau(\mathrm{K})\|f\|_2\left\{\sum_{\nu=m+1}^{\infty}\left\|\sum_{j=0}^{m}\varphi_j\left(\sum_{k=0}^{m}\frac{\varphi_\nu(\pi_{m,k})}{\varphi_j(\pi_{m,k})}\right)-\varphi_\nu\right\|_2^2\right\}^{\frac{1}{2}}.$$

Proof. By *Theorem 3.2.9*, we have

$$\sup_{z\in K}\left|(GPTA/m)_f(z)-f(z)\right|=\sup_{z\in K}\left|T_f(g_m(x,z))-f(z)\right|$$

$$=\sup_{z\in K}\left|\sum_{\nu=0}^{\infty}\left[\sum_{j=0}^{m}\langle f,\varphi_j\rangle\sum_{k=0}^{m}\frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}}-\langle f,\varphi_\nu\rangle\right]\varphi_\nu(z)\right|$$

$$=\sup_{z\in K}\left|\sum_{\nu=0}^{\infty}\left\{\int_\Omega f\left[\sum_{j=0}^{m}\overline{\varphi_j}\sum_{k=0}^{m}\frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}}-\overline{\varphi_\nu}\right]dV\right\}\varphi_\nu(z)\right|$$

$$\le\sup_{z\in K}\left\{\sum_{\nu=0}^{\infty}|\varphi_\nu(z)|^2\right\}^{\frac{1}{2}}\left\{\sum_{\nu=0}^{\infty}\left|\int_\Omega f\,\overline{\sum_{k=0}^{m}\varphi_j\sum_{k=0}^{m}\frac{\varphi_\nu(\pi_{m,k})}{\varphi_j(\pi_{m,k})}-\varphi_\nu}\;dV\right|^2\right\}^{\frac{1}{2}}$$

$$\le\sigma(K)\|f\|_2\left\{\sum_{\nu=0}^{\infty}\left\|\sum_{j=0}^{m}\varphi_j\left[\sum_{k=0}^{m}\frac{\varphi_\nu(\pi_{m,k})}{\varphi_j(\pi_{m,k})}\right]-\varphi_\nu\right\|_2^2\right\}^{\frac{1}{2}},$$

since

$$\sup_{z\in K}\left\{\sum_{\nu=0}^{\infty}|\varphi_\nu(z)|^2\right\}^{\frac{1}{2}}=\sup_{z\in K}\left\{\sum_{\nu=0}^{\infty}\varphi_\nu(z)\overline{\varphi_\nu(z)}\right\}^{\frac{1}{2}}=\sup_{z\in\Omega}\{K_\Omega(z,z)\}^{\frac{1}{2}}=:\sigma(K).$$

This completes the *Proof* of *(a)*. The *Proof* of *(b)* is exactly similar.

Let us now turn to the convergence problem of a generalized Padé-type approximation sequence. From *Corollary 3.2.10*, it follows directly the

Theorem 3.2.11. *Suppose* Ω *is a bounded domain in* $\mathbb{C}^n$ *and* $f\in OL^2(\Omega)$. *Let* $\{\varphi_j(z): j=0,1,2,\ldots\}$ *be any orthonormal basis in* $OL^2(\Omega)$. *Let also*

$$M=(\pi_{m,k})_{m\ge0,0\le k\le m}$$

be an infinite triangular matrix, with elements $\pi_{m,k}$ in

$$\Omega - \left(\bigcup_{0 \le j \le m} Ker \overline{\varphi}_j \right)$$

such that for any $m \ge 0$

$$\pi_{m,k} \ne \pi_{m,k'} \quad (if \;\; k \ne k')$$

and the determinant

$$\det \left[\overline{\varphi_j(\pi_{m,k})} \right]_{k,j} = \begin{vmatrix} \overline{\varphi_0(\pi_{m,0})} & \overline{\varphi_1(\pi_{m,0})} & \dots & \overline{\varphi_m(\pi_{m,0})} \\ \overline{\varphi_0(\pi_{m,1})} & \overline{\varphi_1(\pi_{m,1})} & \dots & \overline{\varphi_m(\pi_{m,1})} \\ \dots\dots\dots & \dots\dots\dots & \dots & \dots\dots\dots \\ \overline{\varphi_0(\pi_{m,m})} & \overline{\varphi_1(\pi_{m,m})} & \dots & \overline{\varphi_m(\pi_{m,m})} \end{vmatrix}$$

is different from zero. If

$$\lim_{m \to \infty} \left\{ \sum_{\nu=0}^{\infty} \left\| \sum_{j=0}^{m} \varphi_j \left(\sum_{k=0}^{m} \frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}} \right) - \varphi_\nu \right\|_2^2 \right\} = 0 ,$$

then the corresponding generalized Padé-type approximation sequence to $f(z)$

$$\left\{ (GPTA/m)_f(z) : m = 0,12,... \right\}$$

converges to $f(z)$ *compactly in* Ω.

Corollary 3.2.12. *Suppose* Ω *is a bounded domain in* $\mathbb{C}^n$ *and* $f(z) \in OL^2(\Omega)$.Let $\{\varphi_j(z) : j = 0,1,2,...\}$ *be any orthonormal basis for* $OL^2(\Omega)$ *and let* M *be an infinite triangular matrix*

$$\mathrm{M} = (\pi_{m,k})_{m \ge 0, 0 \le k \le m}$$

with elements

$$\left(\pi_{m,k}\right) \in \Omega - \left(\bigcup_{0 \le j \le m} Ker\overline{\varphi}_j\right)$$

such that

$$\textit{for any } m \ge 0 \textit{ there holds } \pi_{m,k} \ne \pi_{m,k'} \,(\textit{if } k \ne k')$$

and

$$\{\pi_{m,0}, \pi_{m,1}, \ldots \pi_{m,m}\}$$

satisfies the Haar condition with respect to the Tchebycheff system

$$\{\overline{\varphi_0}, \overline{\varphi_1}, \ldots \overline{\varphi_{m,}}\}.$$

If $\varphi_j \in C\left(\overline{\Omega}\right)$ *for all* $j \ge 0$ *, and*

$$\forall \varepsilon > 0 \; \exists N = N(\varepsilon) : \sup_{x \in \overline{\Omega}} \left| \varphi_i(x) - \sum_{j=0}^{m} \varphi_j(x) \sum_{k=0}^{m} \frac{\varphi_i\left(\pi_{m,k}\right)}{\varphi_j\left(\pi_{m,k}\right)} \right| \langle \varepsilon$$

$\forall m \ge N$ *and* $i \ge 0$ *, then the corresponding generalized Padé-type approximant sequence to* $f(z)$

$$\{(GPTA/m)_f(z) : m = 0,12,\ldots\}$$

converges to $f(z)$ *compactly in* Ω.

It is well known that if Ω is a bounded complete circular domain in $\mathbb{C}^n$, then the monomials

$$\Phi_\Omega = \{\, A_k x^k = A_{k_1,\ldots,k_n} \cdot x_1^{\,k_1} \ldots x_n^{\,k_n} : k = (k_1,\ldots,k_n) \in \mathbb{N}^n\},$$

with

$$A_k := \left(\int_{\Omega} \left| x^k \right|^2 dV(x) \right)^{-\frac{1}{2}},$$

is a complete orthonormal system for $OL^2(\Omega)$. By *Corollary 3.2.12*, we immediately get the following:

Corollary 3.2.13. *Suppose* Ω *is a bounded complete circular domain in* $\mathbb{C}^n$ *and*

$$\mathrm{M} = \{\pi_{m,k} = \pi_{(m_1,\ldots,m_\kappa),(k_1,\ldots,k_n)} : m_j \geq 0,\ 0 \leq \kappa_j \leq m_j \}\}$$

is a sequence with elements

$$\pi_{m,k} \in \Omega - \{(x_1,\ldots,x_n) \in \mathbb{C}^n :\ x_1 \cdots x_n = 0\},$$

such that

$$\textit{for any } m \geq 0 : \pi_{m,k} \neq \pi_{m,k'} \ (\textit{if } k \neq k'),$$

and each finite subsequence

$$\{\pi_{m,k} = \pi_{(m_1,\ldots,m_\kappa),(k_1,\ldots,k_n)} : 0 \leq m_j \leq M_j,\ 0 \leq k_j \leq m_j\ (j = 1,2,\ldots,n)\}$$

satisfies the Haar property with respect to the Tchebycheff system

$$\{A_{m_1,\ldots,m_n} x_1^{m_1} \ldots x_n^{m_n} : 0 \leq m_j \leq (M_j)\ (j = 1,2,\ldots,n)\}.$$

If

$$\lim_{|m| \to \infty} \sup_{i \in \mathrm{N}^n} \left\{ A_i \sup_{x \in \overline{\Omega}} \left| x^i - \sum_{|j| \leq |m|} x^j \sum_{|k| \leq |m|} \pi_{m,k}^{i-j} \right| \right\} = 0,$$

then, for any $f \in OL^2(\Omega)$, *the corresponding generalized Padé-type approximant sequence to* $f(z)$

$$\{(GPTA/m)_f(z) = T_f(g_m(x,z)) : m \in \mathbb{N}^n \ \}$$

converges to $f(z)$ *compactly in* Ω *(as* $|m| \to \infty$).

As we have seen, for any $f \in OL^2(\Omega)$, the functional T_f extends continuously and linearly onto the whole Hilbert space $L^2(\Omega)$. It follows, from the *Riesz Representation Theorem*, that there exists a unique function $F \in L^2(\Omega)$, such that

$$T_f(g) = \int_{\Omega} g(\zeta)\overline{F(\zeta)}\,dV(\zeta)$$

for all $g \in L^2(\Omega)$. If, in particular, $g = \overline{\varphi_\nu}$, then

$$T_f(\overline{\varphi_\nu}) = \int_{\Omega} \overline{\varphi_\nu}\ \overline{F}\ dV = a_\nu^{(f)} = \int_{\Omega} f\ \overline{\varphi_\nu}\ dV\ ,$$

and therefore

$$\int_{\Omega} \left[f - \overline{F}\right]\overline{\varphi_\nu}\ dV = 0\text{, for any } \nu = 0,1,2,...$$

This means that the function $\left(f - \overline{F}\right)$ is orthogonal to every φ_ν. By completeness of the basis $\{\varphi_\nu : \nu = 0,1,2,...\}$, we conclude that

$$f = \overline{F}\ .$$

Hence

$$T_f(g) = \int_{\Omega} g\ f\ dV \qquad (g \in L^2(\Omega)).$$

Theorem 3.2.18. *Every generalized Padé-type approximant* $(GPTA/m)_f(z)$ to $f \in OL^2(\Omega)$, *has the following integral representation*

$$(GPTA/m)_f(z) = \int_{\Omega} f(x)\ D_m(x,z)\ dV(x)\ ,$$

where the kernel $D_m(x,z)$ *equals*

$$D_m(x,z)=\sum_{k=0}^{m}K_\Omega(z,\pi_{m,k})\sum_{j=0}^{m}\frac{\overline{\varphi_j(x)}}{\varphi_j(\pi_{m,k})}.$$

Proof. It holds

$$\begin{aligned}(GPTA/m)_f(z)&=T_f(g_m(x,z))\\&=\int_\Omega g_m(x,z)\,f(x)\,dV(x)=\int_\Omega\sum_{j=0}^{m}c_j^{(m)}(z)\,\overline{\varphi_j(x)}\,f(x)\,dV(x)\\&=\int_\Omega f(x)\,\sum_{j=0}^{m}\overline{\varphi_j(x)}\sum_{k=0}^{m}\frac{K_\Omega(z,\pi_{m,k})}{\varphi_j(\pi_{m,k})}\,dV(x)\\&=\int_\Omega f(x)\left[\sum_{k=0}^{m}K_\Omega(z,\pi_{m,k})\sum_{j=0}^{m}\frac{\overline{\varphi_j(x)}}{\varphi_j(\pi_{m,k})}\right]dV(x).\end{aligned}$$

By *Theorem 3.2.8.(a)*, $T_f(g_m(x,\cdot))\in OL^2(\Omega)$, for any $f\in OL^2(\Omega)$. It follows, from the *Closed Graph Theorem*, that the integral linear operator

$$T_f(g_m(x,\cdot)):OL^2(\Omega)\to OL^2(\Omega):f(z)\mapsto T_f(g_m(x,z))\equiv\int_\Omega f(x)\,D_m(x,z)\,dV(x)$$

is continuous. We call this operator the *generalized Padé-type operator for* $OL^2(\Omega)$. Its adjoint is the operator

$$T_f(g_m(x,\cdot))^*:OL^2(\Omega)\to OL^2(\Omega):h(z)\mapsto T_h(g_m(x,z))^*\equiv\int_\Omega h(x)\,\overline{D_m(x,z)}\,dV(x).$$

In fact, to $T_f(g_m(x,\cdot))$ there corresponds a unique operator

$$T_f(g_m(x,\cdot))^*:OL^2(\Omega)\to OL^2(\Omega)$$

satisfying

$$\int_\Omega T_f(g_m(x,z))\cdot\overline{h(z)}dV(z)=\langle T_f(g_m(x,\cdot)),\,h\rangle_2=\langle f,T_h(g_m(x,\cdot))^*\rangle_2$$

$$= \int_{\Omega} f(z)\, \overline{T_h(g_m(x,z))^*}\, dV(z)$$

for all $f \in OL^2(\Omega)$ and $h \in OL^2(\Omega)$. Since, by *Fubini's Theorem*:

$$\begin{aligned}\int_{\Omega} T_f(g_m(x,z))\, \overline{h(z)}\, dV(z) &= \int_{\Omega} \int_{\Omega} f(x)\, D_m(x,z)\, dV(x)\, \overline{h(z)}\, dV(z)\\ &= \int_{\Omega} f(x) \left[\overline{\int_{\Omega} h(z) \overline{D_m(x,z)}\, dV(z)} \right] dV(x)\\ &= \int_{\Omega} f(z) \left[\overline{\int_{\Omega} h(x) \overline{D_m(z,x)}\, dV(x)} \right] dV(z),\end{aligned}$$

it is immediately verified that

$$T_h(g_m(x,z))^* = \int_{\Omega} h(x)\, \overline{D_m(z,x)}\, dV(x).$$

The continuity of the generalized Padé-type operator leads to some interesting convergence results.

Theorem 3.2.19. *If the sequence* $\{f_\nu \in OL^2(\Omega):\ \nu = 0,1,2,\ldots\}$ *converges to the function* $f \in OL^2(\Omega)$ *in the* L^2-norm, *then*

$$\lim_{\nu\to\infty} (GPTA/m)_{f_\nu}(z) = (GPTA/m)_f(z)$$

in the L^2-norm.

Corollary 3.2.20. *If the series of functions*

$$\sum_{\nu=0}^{\infty} c_\nu\, f_\nu(z) \quad (c_\nu \in \mathbb{C},\ f \in OL^2(\Omega))$$

converges to $f(z) \in OL^2(\Omega)$ *in the* L^2-norm*, then*

$$(GPTA/m)_f(z) = \sum_{\nu=0}^{\infty} c_\nu (GPTA/m)_{f_\nu}(z)$$

in the L^2-norm.

In *Section 3.4*, we will see that all the above ideas extend to the context of a functional Hilbert space. This theoretical approach will permit us to establish more general results having useful applications to several concrete examples.

3.3. The Continuous Case

3.3.1. Markov's Inequalities

Our next purpose is to introduce generalized Padé-type approximation to continuous functions of several complex variables. Our theoretical method leads to aspects and results extending the analytic L^2-case and will require the validity of Markov's inequalities into a compact subset of $\mathbb{C}^n$. We therefore will discuss these first.

The classical Markov's inequalities in the closed unit cube $\mathrm{I}^n = [-1,1]^n \subset \mathbb{R}^n$ permit us to estimate the growth of successive derivatives of a polynomial by its degree and uniform norm in I^n:

$$\left\|D^a Q\right\|_{\mathrm{I}^n} \leq (\deg Q)^{2|a|} \cdot \left\|Q\right\|_{\mathrm{I}^n}, \text{ for any } Q \in \mathbb{P}(\mathbb{R}^n) \text{ and any } a = (a_1,...,a_n) \in \mathbb{N}^n.$$

Here $\mathbb{P}(\mathbb{R}^n)$ is the space of all complex polynomials in $\mathbb{R}^n$, $\|Q\|_{I^n}$ equals the $\sup_{x\in I^n}|Q(x)|$ and $|a|=a_1+...+a_n$. The extension of these inequalities to more general families of compact subsets of $\mathbb{R}^n$ inspired many people. A very detailed survey of this work can be founded in [5], [116], [117], [118] and [127].

In what follows, we will consider $\mathbb{R}^n$ as a subspace of $\mathbb{C}^n$. We shall say that the *Markov inequality* (M_∞) *is true on a compact subset* E of $\mathbb{C}^n$, if there exists an integer $m \geq 1$ such that

(M_∞) $\qquad \left\|D^a Q\right\|_E \leq M_a(\deg Q)^{m|a|}\cdot\|Q\|_E$ for any $Q\in\mathbb{P}(\mathbb{C}^n)$ and any $a\in\mathbb{N}^n$.

The constant M_a depends on a, but is independent of Q. Surely, the most far-reaching program to extend Markov's inequality (M_∞) to more general families of compact sets has been initialized by Pawlucki and Plésniak in [116]. The single most important contribution to come from this program is the discovery of the central role of *uniformly polynomially cuspidal subsets* of R^n or $\mathbb{C}^n$. A subset E of K^n (K=R or $\mathbb{C}$) is uniformly polynomially cuspidal, if there exist positive constants M and m, and a positive integer d such that for each point $x\in\overline{E}$, one may choose a polynomial map $h_x:\mathrm{K}\to\mathrm{K}^n$ of degree at most d satisfying the following conditions:

$$h_x([0,1])\subset E \text{ and } h_x(0)=x,$$

and

$$dist(h_x(t),\mathbb{C}^n-E)\geq M\,t^m \quad \textit{for all } x\in\overline{E} \textit{ and all } t\in[0,1].$$

One can verify that every bounded convex with non void interior or bounded Lipschitz domain in K^n is uniformly polynomially cuspidal $([5])$. Further, an application of *Hironaka's Rectilinearization Theorem* and *Lojasiewicz's Inequality* shows that every bounded subanalytic subset E of $\mathbb{C}^n$ such that the interior $\operatorname{int}E$ of E is dense in E, is uniformly polynomially cuspidal. Following [116], we have the

Theorem 3.3.1. *The Markov inequality* (M_∞) *is true into every uniformly polynomially cuspidal compact subset of* K^n.

We also give a general definition for a measure on a compact set to satisfy an L^p-Markov inequality: given a compact set E in K^n and a positive measure μ on E, we say that (E,μ) *satisfies the Markov inequality* (M_p) for some $p>0$, if for any $a\in\mathbb{N}^n$ there are a constant $M=M(p,a)$ and two positive integers $r,m>1$, such that

$$(\mathrm{M}_p) \qquad \left\|D^a Q\right\|_E \leq M(\deg Q)^{\frac{m}{p}+r|a|}\left(\int_E |Q|^P\, d\mu\right)^{\frac{1}{p}} \quad \textit{for any } Q\in \mathbb{P}(\mathrm{K}^n).$$

If E fulfills (M_∞), then a sufficient condition for (E,μ) to satisfy (M_p) is

$$\|Q\|_E \leq M(\deg Q)^{\frac{m}{p}}\left(\int_E |Q|^P\, d\mu\right)^{\frac{1}{p}} \quad \textit{for any } Q\in \mathbb{P}(\mathrm{K}^n).$$

In [147], Zeriahi proved that if E is a uniformly polynomially cuspidal compact subset of $\mathbb{C}^n$ and if μ is the Lebesgue measure on E, then for any $p>0$ there holds

$$\|Q\|_E \leq M(N\deg Q)^{\frac{m}{p}}\left(\int_E |Q|^P\, d\mu\right)^{\frac{1}{p}} \quad (Q\in \mathbb{P}(\mathbb{C}^n)),$$

where M,N and m are positive constants independent of Q and p.

One consequence of Markov's property (M_2) for (E,μ) is that the vector space $C^\infty(E)$, of all complex-valued functions defined on a compact set E in R^n and admitting a C^∞ extension on R^n, has a Schauder basis consisting of orthogonal polynomials. To see this, let $\nu:\mathbb{N}\to\mathbb{N}^n$ be a bijection with $|\nu(j)|\leq|\nu(j+1)|$ for any j. If (E,μ) satisfies (M_2), the set

$$\{x^{\nu(j)} : j = 0,1,2,...\}$$

is linearly independent in $L^2(E,\mu)$ and, by the *Hilbert – Schmidt Orthogonalization Process*, one can construct a family

$$\{\varphi_j : j = 0,1,2,...\}$$

of orthonormal polynomials in $L^2(E,\mu)$, such that $\deg\varphi_j = |\nu(j)|$, $j = 0,1,2,...$ For each $u \in L^2(E,\mu)$, we then write

$$c_j(u) := \int_E u\,\overline{\varphi_j}\,d\mu \quad (j = 0,1,2,...).$$

Theorem 3.3.2.([147]). *If* $u \in C^\infty(E)$, *then there holds*

$$u(z) = \sum_{j=0}^{\infty} c_j(u)\,\varphi_j(z) \quad \textit{uniformly on } E.$$

Proof. Let $u \in C^\infty(E)$. By the orthonormality of the system $\{\varphi_j : j = 0,1,2,...\}$, we get

$$c_j(u) = \int_E (u - Q)\,\overline{\varphi_j}\,d\mu \qquad j = 0,1,2,...,$$

for any $Q \in \mathrm{P}(\mathbb{C}^n)$ with $\deg Q < \deg\varphi_j$. Application of *Cauchy – Schwarz's inequality* shows that

$$|c(u)| \le \inf\ \{\ \|u - Q\|_E = \left(\int_E |u - Q|^2 d\mu\right)^{\frac{1}{2}} : Q \in \mathrm{P}(\mathbb{C}^n) \textit{ and } \deg Q < \deg\varphi_j\ \}$$

or alternatively

$$|c_j(u)| \le (\mu(E))^{\frac{1}{2}}\ \inf\{\|u - Q\|_E : Q \in \mathrm{P}(\mathbb{C}^n) \textit{ and } \deg Q < \deg\varphi_j\ \}\ (j = 0,1,2,...).$$

Put

$$p_j(u) := \inf\{\|u - Q\|_E : Q \in \mathrm{P}(\mathbb{C}^n) \textit{ and } \deg Q < \deg\varphi_j\ \}.$$

Since, by *Jackson's Theorem*, the sequence $\{p_j(u) : j = 0,1,2,...\}$ is rapidly decreasing, the above estimate implies that

$$\lim_{j\to\infty} (\deg \varphi_j)^k |c_j(u)| = 0 \qquad \text{for any } k \geq 0 .$$

As (E, μ) satisfies Markov's inequality (M_2), we also have

$$\|D^a \varphi_j\|_E \leq (M \deg \varphi_j)^{m+r|a|} \qquad \text{for any } j \geq 0 \text{ and } a \in \mathbb{N}^n,$$

which in particular gives

$$|c_j(u)| \, \|\varphi_j\|_E \leq M \, | \, c_j(u) \, | \deg \varphi_j{}^m \quad (j \geq 0).$$

Since $(\deg \varphi_j) \sim j^{\frac{1}{n}}$, the *Proof* of the *Theorem* is achieved.

The results we have obtained generalize to the context of a compact set $E \subset \mathrm{R}^n$ satisfying (M_∞):

$$\|D^a Q\|_E \leq M_a (\deg Q)^{m \cdot |a|} \cdot \|Q\|_E \qquad (Q \in \mathbb{P}(\mathrm{C}^n) \text{ and } a \in \mathbb{N}^n).$$

To prove this, we may first define a nuclear Fréchet topology on $\mathbb{P}(\mathrm{R}^n)$, by introducing seminorms $q_{\mathrm{K},\lambda}$ on $C^\infty(E)$:

$$q_{\mathrm{K},\lambda}(u) = \inf \{ \sup_{|a| \leq \lambda} \|D^a g\|_{\mathrm{K}} : g \in C^\infty(\mathrm{R}^n), \; g / E = u \}$$

$(u \in C^\infty(E), \; \mathrm{K} \subset\subset \mathrm{R}^n, \; \lambda \in \mathbb{N})$.

By (M_∞), if $Q \in \mathbb{P}(\mathrm{C}^n)$ and $Q / E = 0$, then $Q \equiv 0$. From *Jackson's Theorem*, it therefore follows that for any $g \in C^{\infty}(\mathrm{R}^n)$, such that $g/E = 0$, the restriction of any derivative of g to E is equal to 0. This means that the injective restriction $C^{\infty}(E) \to C(E)$ is continuous. Since $C^{\infty}(E)$ is a nuclear Fréchet space, *Mityagin's Theorem* ([106]) guarantees the existence of a Hilbert space H such that the injections

$$C^{\infty}(E) \to H \to C(E)$$

are continuous.

Let now again $\nu : \mathbb{N} \to \mathbb{N}^n$, be a bijection with $|\nu(j)| \leq |\nu(j+1)|$ for any j. Since E satisfies (M_{∞}), the set $\{x^{\nu(j)} : j = 0,1,2,...\}$ is linearly independent in H and hence, by the *Hilbert-Schmidt Orthogonalization Process*, one can find a system

$$\{\psi_j : j = 01,2,...\} \subset H,$$

consisting of orthonormal polynomials with $\deg \psi_j = |\nu(j)|$ for any j.

From the continuity of the applications $C^{\infty}(E) \to H \to C(E)$, we then get the existence of a constant $B > 0$ such that $\|\psi_j\|_E \leq B$ for all j. It follows, from (M_{∞}), that

$$\|D^a \psi_j\|_E \leq B\, M_a \left(\deg \psi_j\right)^{m|a|} \qquad \text{for any } j \geq 0 \text{ and } a \in \mathbb{N}^n.$$

On the other hand, the continuity of the injection $C^{\infty}(E) \to H$ shows that if $E \subset \mathrm{I}^n$, where I is a closed interval of $\mathbb{R}$, then there exists a constant A and an integer $\lambda \geq 1$ such that $\|u\|_H \leq A\, q_{\mathrm{I}^n, \lambda}(u)$, whenever $u \in C^{\infty}(E)$.

Let $\tilde{u}$ be a C^{∞} extension of a $u \in C^{\infty}(E)$ on R^n. For each j, let Q_j be the polynomial of $\mathbb{P}(\mathrm{R}^n)$, with degree $\deg Q_j \leq j$ and fulfilling

$$\rho_j\left(\widetilde{u},\mathrm{I}^n\right):=\inf\left\{\ \left\|\widetilde{u}-Q\right\|_{\mathrm{I}^n}:Q\in\mathbb{P}(\mathbb{C}^n)\text{ and }\deg Q_j\le j\ \right\}=\left\|\widetilde{u}-Q_j\right\|_{\mathrm{I}^n}.$$

If $\langle\cdot/\cdot\rangle_H$ is a notation for the inner product of H with corresponding norm $\|\cdot\|_H$, we put

$$c_j(u):=\langle u/\psi_j\rangle_H\qquad(j=0,1,2,...).$$

By the orthogonality property of the basis $\{\psi_j: j\ge 0\}$, we can write $c_j(u)=\langle u-Q_{s_j}/\psi_j\rangle_H$ for any $j\ge 1$. It follows from *Cauchy-Schwarz's Inequality* that

$$\left|c_j(u)\right|\le A\sup_{|a|\le\lambda}\left\|D^a(\widetilde{u}-Q)\right\|_{\mathrm{I}^n}\qquad(j\ge 0).$$

Set now

$$P_0:=Q_0\qquad\text{and}\qquad P_j:=Q_j-Q_{j-1}\quad(j\ge 1).$$

Since $\widetilde{u}\in C^\infty\left(\mathrm{I}^n\right)$, the sequence $\{\rho_j(\widetilde{u},\mathrm{I}^n): j=0,1,2,...\}$ is rapidly decreasing. This implies that for any $a\in\mathbb{N}^n$ there holds

$$D^a(\widetilde{u})=\sum_{j=0}^{\infty}D^aP_j,\qquad\text{in }C^\infty\left(\mathrm{I}^n\right),$$

and consequently,

$$\left\|D^a\left(\widetilde{u}-Q_j\right)\right\|_{\mathrm{I}^\nu}\le\sum_{k=j+1}^{\infty}\left\|D^aP_k\right\|_{\mathrm{I}^n}.$$

But, from Markov's inequality (M_∞) in the cube I^n, we have

$$\left\|D^aP_k\right\|_{\mathrm{I}^n}\le C_ak^{2|a|}\left\|P_k\right\|_{\mathrm{I}^n}\le C_ak^{2|a|}\left(\left\|\ \widetilde{u}\ -Q_k\right\|_{\mathrm{I}^n}+\left\|\ u-Q_{k-1}\right\|_{\mathrm{I}^n}\right).$$

Thus,

$$\left\|D^a\left(\widetilde{u}-Q_j\right)\right\|_{\mathrm{I}^n}\le 2M_a\sum_{k=j+1}^{\infty}k^{2|a|}\ \rho_{k-1}\left(\widetilde{u},\mathrm{I}^n\right).$$

The rapid decrease of the sequence $\{\rho_j(\widetilde{u},\mathrm{I}^n): j=0,1,2,...\}$ now guarantees that, for any

$N > 0$, the supremum

$$A_N = \sup_{k \geq 0, |a| \leq \lambda} \left\{ 2M_a k^{2|a|+N+2} \rho_{k-1}\left(\tilde{u}, \mathrm{I}^n\right) \right\}$$

is a finite number. We infer that

$$\left(\deg \psi_j\right)^N \left| c_j(u) \right| \leq (A\, A_N) \sum_{k=s_j}^{\infty} k^{-2} \qquad (j \geq 0, N > 0),$$

and hence $\lim_{j \to \infty} (\deg \psi_j)^N \mid c_j(u) \mid = 0 \qquad (N > 0)$. Since $\left(\deg \psi_j\right) \sim j^{\frac{1}{2}}$, we have thus proved the following

Theorem 3.3.3.([147]) *If* E *satisfies Markov's inequality* (M_∞), *then there holds*

$$u(z) = \sum_{j=0}^{\infty} c_j(u) \psi_j(z), \text{ uniformly on } E,$$

whenever $u \in C^\infty(E)$.

3.3.2. Generalized Padé-type Approximants to Continuous Functions

Let E be a compact subset of R^n, ($E \neq \varnothing$).Suppose μ is a positive measure on E and assume that (E, μ) satisfies Markov's inequality (M_2). In this *Paragraph*, we will define generalized Padé-type approximants to continuous functions on E.

As we have already seen , there is a family $\{\varphi_j : j = 0,1,2,...\}$ of orthonormal polynomials in $L^2(E, \mu)$, such that $\left|\deg \varphi_j\right| \leq \left|\deg \varphi_{j+1}\right| \quad (j = 0,1,2,...)$ and every $u \in C^\infty(E)$ can be written as

$$u(z)=\sum_{j=0}^{\infty}c_j(u)\,\varphi_j(z)$$

where

$$c_j(u)=\int_E u\,\overline{\varphi}_j\,d\mu \qquad (j=0,1,2,...)$$

and where the series converges uniformly on E.

In the sequel, we shall assume that $\{\varphi_j : j=0,1,...\}$ is a *self-summable family* in $L^2(E,\mu)$, i.e. for any $z\in E$, the sequence $\{\varphi_j(z)\overline{\varphi_j(z)} : j=0,1,2,...\}$ is summable in $L^2(E,\mu)$. This means that for every $z\in E$ and every positive number ε there exists a finite set $J_0=J_0(z,\varepsilon)$ of indices such that

$$\left\|\sum_{j\in J}\varphi_j(z)\overline{\varphi_j}\right\|_E := \left(\int_E\left|\sum_{j\in J}\varphi_j(z)\overline{\varphi_j}\right|^2 d\mu\right)^{1/2} < \varepsilon\,,$$

whenever J is a finite set of indices disjoint from J_0 ([74]). By this summability condition, for each $z\in E$ fixed, the function

$$\mathrm{K}_E^{(2)}(z,\cdot):E\to\mathbb{C}\cup\{\infty\}:x\mapsto \mathrm{K}_E^{(2)}(z,x):=\sum_{j=0}^{\infty}\varphi_j(z)\overline{\varphi_j(x)}$$

is in $L^2(E,\mu)$.

Let now $u\in C^{\infty}(E)$. We introduce the linear functional

$$T_u^{(\mu)}:\overline{\Phi(\mathbb{C}^n)}\to\mathbb{C}:\overline{\varphi_j(x)}\mapsto T_u^{(\mu)}\left(\overline{\varphi_j(x)}\right):=c_j(u),$$

where $\overline{\Phi(\mathbb{C}^n)}$ is the complex vector space which is spanned by all finite complex combinations of $\overline{\varphi_j}$'s. If

$$p(x) = \sum_{\nu=0}^{m} \beta_\nu \overline{\varphi_\nu(x)} \in \overline{\Phi(\mathbb{C}^n)},$$

then

$$\left|T_u^{(\mu)}(p(x))\right| = \left|T_u^{(\mu)}\left(\sum_{\nu=0}^{m} \beta_\nu \overline{\varphi_\nu(x)}\right)\right| = \left|\sum_{\nu=0}^{m} \beta_\nu T_u^{(\mu)}\left(\overline{\varphi_\nu(x)}\right)\right|$$

$$= \left|\sum_{\nu=0}^{m} \beta_\nu c_\nu(u)\right| = \left|\sum_{\nu=0}^{m} \beta_\nu \int_E u \overline{\varphi_\nu} \, d\mu\right|$$

$$= \left|\int_E u \left(\sum_{\nu=0}^{m} \beta_\nu \overline{\varphi_\nu}\right) d\mu\right| = \left|\int_E u \, p \, d\mu\right|.$$

From *Hölder's Inequality*, it follows that

$$\left|T_u^{(\mu)}(p)\right| \le \left(\int_E |u|^2 d\mu\right)^{\frac{1}{2}} \left(\int_E |p|^2 d\mu\right)^{\frac{1}{2}} = \|u\|_E \, \|p\|_E,$$

and, by the $Hahn - Banach\ Theorem$, $T_u^{(\mu)}$ extends to a linear continuous functional on $L^2(E,\mu)$. For each $z \in E$ fixed, one can therefore define the number

$$T_u^{(\mu)}\left(K_E^{(2)}(z,x)\right),$$

where $T_u^{(\mu)}$ acts on the variable $x \in E$. Furthermore, by continuity, we get

$$u(z) = \sum_{j=0}^{\infty} c_j(u)\varphi_j(z) = \sum_{j=0}^{\infty} T_u^{(\mu)}\left(\overline{\varphi_j(x)}\varphi_j(z)\right) = T_u^{(\mu)}\left(\sum_{j=0}^{\infty} \overline{\varphi_j(x)}\varphi_j(z)\right) = T_u^{(\mu)}\left(K_E^{(2)}(z,x)\right).$$

Thus, computing $u(z)$ for a fixed value of $z \in E$ is nothing else than computing

$$T_u^{(\mu)}\left(K_E^{(2)}(z,x)\right).$$

If only a few Fourier coefficients $c_j(u)$ of u are known, then the function $K_E^{(2)}(z,x)$ has to be replaced by a simpler expression.

For any $m = 0,1,2,\ldots$, let us consider the $(m+1)-$dimensional complex vector space

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$$\overline{F_{m+1}}$$

spanned by the Tchebycheff system $\{\overline{\varphi_0}, \overline{\varphi_1}, \ldots \overline{\varphi_{m,}}\}$, and suppose that $\overline{F_{m+1}}$ satisfies the Haar condition into a finite set of pair-wise distinct points

$$\mathrm{M}_{m+1} = \{\pi_{m,0}, \pi_{m,1}, \ldots \pi_{m,m}\} \subset E,$$

with

$$\mathrm{M}_{m+1} \cap \left(\bigcup_{0 \le j \le m} Ker\overline{\varphi_j} \right) = \varnothing \text{ (}: Ker\overline{\varphi_j} \text{ is the zero set of } \overline{\varphi_j}\text{).}$$

In other words, suppose that every function in $\overline{F_{m+1}}$ has at most m roots in M_{m+1}. By *Theorem 3.2.5*, for any $z \in E$ there is a unique

$$g_m(x,z) = \sum_{j=0}^{\infty} \sigma_j^{(m)}(u) \overline{\varphi_j(x)} \in \overline{F_{m+1}}$$

satisfying

$$g_m(x, \pi_{m,k}) = \sum_{j=0}^{\infty} \sigma_j^{(m)}(u) \overline{\varphi_j(\pi_{m,k})} = K_E^{(2)}(z, \pi_{m,k}), \quad \text{for any } k \le m.$$

A necessary and sufficient condition for the existence of a unique solution $\left(\sigma_0^{(m)}(z), \sigma_1^{(m)}(z), \ldots, \sigma_m^{(m)}(z)\right)$ for this linear system is that the determinant

$$\det\left[\overline{\varphi_j(\pi_{m,k})}\right]_{k,j} := \begin{vmatrix} \overline{\varphi_0(\pi_{m,0})} & \overline{\varphi_1(\pi_{m,0})} & \ldots & \overline{\varphi_m(\pi_{m,0})} \\ \overline{\varphi_0(\pi_{m,1})} & \overline{\varphi_1(\pi_{m,1})} & \ldots & \overline{\varphi_m(\pi_{m,1})} \\ \cdots\cdots & \cdots\cdots & \ldots & \cdots\cdots \\ \overline{\varphi_0(\pi_{m,m})} & \overline{\varphi_1(\pi_{m,m})} & \ldots & \overline{\varphi_m(\pi_{m,m})} \end{vmatrix}$$

is different from zero. Notice that this condition is equivalent to the Haar condition for $\overline{F_{m+1}}$ into the set M_{m+1}. Then, for any $j = 0,1,2,\ldots,m$, there holds

$$(Q_j) \qquad \sigma_j^{(m)}(z) = \sum_{k=0}^{m} \frac{K_E^{(2)}(z, \pi_{m,k})}{\varphi_j(\pi_{m,k})}.$$

Definition 3.3.4. *Let $E \subset\subset \mathbb{C}^n$ and let μ be a positive measure of E, such that (E, μ) satisfies (M_2). Assume that $\{\varphi_j : j = 0,1,2,\ldots\}$ is a self - summable family, consisting of orthonormal polynomials in $L^2(E, \mu)$ such that $|\deg \varphi_j| \le |\deg \varphi_{j+1}|$ ($j = 0,1,2,\ldots,$). For $m \ge 0$, choose a finite set of pair-wise distinct points*

$$\mathrm{M}_{m+1} = \{\pi_{m,0}, \pi_{m,1}, \ldots \pi_{m,m}\} \subset E - \left(\bigcup_{0 \le j \le m} Ker\overline{\varphi_j} \right),$$

so that

$$\det\left[\overline{\varphi_j(\pi_{m,k})}\right]_{k,j} \neq 0$$

and for any $k \le m$ the series

$$\sum_{j=0}^{\infty} \varphi_j(\cdot)\overline{\varphi_j(\pi_{m,k})}$$

converges uniformly on E. Any function $(GPTA/m)_u^{(\mu)}(z)$, defined by

$$T_u^{(\mu)}(g_m(x,\cdot)) : E \to \mathbb{C} \;\; z \mapsto (GPTA/m)_u^{(\mu)}(z) := T_u^{(\mu)}(g_m(x,z))$$

is called a generalized Padé-type approximant to $u \in C^\infty(E)$, *with generating system M_{m+1}.*

If, moreover, for every $\nu = 0,1,2,\ldots,m$, there holds

$$\sum_{\substack{j=0 \\ (j \neq \nu)}}^{m} c_j(u) \sum_{k=0}^{m} \frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}} = 0,$$

then the function $T_u^{(\mu)}(g_m(x,\cdot))$ is said to be a Padé-type approximant to u, with generating system M_{m+1} .It is denoted by $(PTA/m)_u^{(\mu)}(z)$.

This *Definition* seems be similar to the corresponding one, listed in *Paragraph 3.2.3*. The only difference consists in the supplementary presupposition about uniform convergence for the series

$$\sum_{j=0}^{\infty} \varphi_j(\cdot)\overline{\varphi_j(\pi_{m,k})}.$$

This presupposition guarantees that $K_E^{(2)}(\cdot,\pi_{m,\kappa}) \in C^\infty(E) \quad (k = 0,1,2,...,m)$, and therefore that the generalized Padé-type approximant $T_u^{(\mu)}(g_m(\cdot,x))$ is a continuous function on E: in fact, by (Q_j), there holds

$$T_u^{(\mu)}(g_m(x,z)) = T_u^{(\mu)}\left(\sum_{j=0}^{m} \sigma_j^{(m)}(z)\overline{\varphi_j(x)}\right)$$

$$= \sum_{j=0}^{m} c_j(u)\sigma_j^{(m)}(z) = \sum_{j=0}^{m} c_j(u)\sum_{j=0}^{m} \frac{K_E^{(2)}(z,\pi_{m,k})}{\varphi_j(\pi_{m,k})} \in C^\infty(E).$$

Notice that the computation of a generalized Padé-type approximant $T_u^{(\mu)}(g_m(x,z))$ to $u \in C^\infty(E)$ requires only the knowledge of the Fourier coefficients

$$c_0(u), c_1(u), ..., c_m(u)$$

of u and of the functions

$$\sigma_0^{(m)}(z), \sigma_1^{(m)}(z), ..., \sigma_m^{(m)}(z),$$

resulting from the equations $(Q_0), (Q_1), ..., (Q_m)$ respectively.

Under the assumptions of *Definition 3.3.4*, we also have a direct analogous to *Theorem 3.2.8*, which justifies the notation Padé-type approximant :

Theorem 3.3.5. *If*

$$\sum_{\nu=0}^{\infty} \beta_\nu^{(m,u)} \varphi_\nu(z)$$

is the Fourier expansion of a Padé-type approximant $(PTA/m)_u^{(\mu)}(z) \equiv T_u^{(\mu)}(g_m(x,z))$ *to*

$$u(z) = \sum_{\nu=0}^{\infty} c_\nu(u)\, \varphi_\nu(z) \in C^\infty(E),$$

with respect to the family $\{\varphi_\nu : \nu = 0,1,2,...\}$, *then*

$$\beta_\nu^{(m,u)} = c_\nu(u), \quad \textit{for every } \nu = 0,1,2,...,m.$$

Proof. Since $K_E^{(2)}(\cdot, \pi_{m,k}) \in C^\infty(E)$ for any $k \le m$, each function

$$\sigma_j^{(m)}(z) = \sum_{k=0}^{m} \frac{K_E^{(2)}(z, \pi_{m,k})}{\overline{\varphi_j(\pi_{m,k})}}$$

is continuous on E $(j = 0,1,2,...,m)$. From *Corollary 3.3.3*, it follows that there are Fourier coefficients $t_\nu^{(j,m)}$, such that

$$\sigma_j^{(m)}(z) = \sum_{\nu=0}^{\infty} t_\nu^{(j,m)} \varphi_\nu(z), \quad \text{uniformly on } E.$$

We can therefore write

$$T_u^{(\mu)}(g_m(x,z)) = \sum_{j=0}^{m} c_j(u) \sigma_j^{(m)}(z) = \sum_{j=0}^{m} c_j(u) \sum_{\nu=0}^{\infty} t_\nu^{(j,m)} \varphi_\nu(z) = \sum_{\nu=0}^{\infty} \left(\sum_{j=0}^{m} t_\nu^{(j,m)} c_j(u) \right) \varphi_\nu(z).$$

This implies that

$$\beta_\nu^{(m,u)} = \sum_{j=0}^{m} t_\nu^{(j,m)} c_j(u)$$

and alternatively,

$$\begin{aligned}\beta_\nu^{(m,u)} &= \sum_{j=0}^{m}\left(\int_E \sigma_j^{(m)}(\zeta)\overline{\varphi_\nu(\zeta)}\,d\mu(\zeta)\right)c_j(u)\\ &= \sum_{j=0}^{m}\left(\int_E \sum_{k=0}^{m}\frac{K_E^{(2)}(\zeta,\pi_{m,k})}{\varphi_j(\pi_{m,k})}\overline{\varphi_\nu(\zeta)}\,d\mu(\zeta)\right)c_j(u)\\ &= \sum_{j=0}^{m}\sum_{k=0}^{m}\frac{1}{\varphi_j(\pi_{m,k})}\left(\int_E K_E^{(2)}(\zeta,\pi_{m,k})\overline{\varphi_\nu(\zeta)}\,d\mu(\zeta)\right)c_j(u).\end{aligned}$$

Now observe that the series

$$\sum_{\nu=0}^{\infty}\varphi_\nu(\zeta)\overline{\varphi_\nu(\pi_{m,k})}$$

converges to the continuous function $K_E^{(2)}(\zeta,\pi_{m,k})$ uniformly on E. Hence, by orthonormality, we obtain

$$\int_E K_E^{(2)}(\zeta,\pi_{m,k})\overline{\varphi_\nu(\zeta)}d\mu(\zeta)=\overline{\varphi_\nu(\pi_{m,k})} \qquad (k\le m).$$

Thus, from the definition of Padé-type approximants, it follows that for any $\nu=0,1,2,\ldots,m$ there holds

$$\beta_\nu^{(m,u)}=\sum_{j=0}^{m}\sum_{k=0}^{m}\frac{\overline{\varphi_\nu(\pi_{m,k})}}{\varphi_j(\pi_{m,k})}c_j(u)=c_\nu(u)+\sum_{\substack{j=0\\(j\ne\nu)}}^{m}\sum_{k=0}^{m}\frac{\overline{\varphi_\nu(\pi_{m,k})}}{\varphi_j(\pi_{m,k})}c_j(u)=c_\nu(u),$$

which completes the *Proof* of the *Theorem*.

Repetition of the *Proof* of *Theorem 3.2.9*, with only formal changes, gives the error formulas.

Theorem 3.3.6. *The error of a generalized Padé-type approximation equals*

$$T_u^{\{\mu\}}(g_m(x,z))-u(z)=\sum_{\nu=0}^{\infty}\left[\sum_{\substack{j=0\\(j\ne\nu)}}^{m}c_j(u)\sum_{k=0}^{m}\frac{\overline{\varphi_\nu(\pi_{m,k})}}{\varphi_j(\pi_{m,k})}\right]\varphi_\nu(z);$$

the error of a Padé-type approximation is

$$T_u^{\{\mu\}}\left(g_m(x,z)\right)-u(z)=\sum_{\nu=m+1}^{\infty}\left[\sum_{\substack{j=0\\(j\neq\nu)}}^{m}c_j(u)\sum_{k=0}^{m}\frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}}\right]\varphi_\nu(z).$$

Let us give a different approach to the generalized Padé-type approximation on a compact subset E of R^n ($E\neq\varnothing$) satisfying Markov's property (M_2) with respect to some positive measure μ.

For $u\in C^\infty(E)$, the corresponding linear functional $T_u^{(\mu)}$ extends continuously and linearly onto the Hilbert space $L^2(E,\mu)$. By *Riez's Representation Theorem*, there exists a unique element $U\in L^2(E,\mu)$ satisfying

$$T_u^{(\mu)}(g)=\int_E g\,\overline{U}d\mu,\quad \text{whenever } g\in L^2(E,\mu).$$

For $g=\overline{\varphi_\nu}$, we therefore obtain

$$T_u^{(\mu)}\left(\overline{\varphi_\nu}\right)=\int_E\overline{\varphi_\nu}\;\overline{U}\;d\mu=c_\nu(u)=\int_E u\,\overline{\varphi_\nu}\;d\mu$$

and, consequently

$$\int_E\left[u-\overline{U}\right]\overline{\varphi_\nu}d\mu=0,\qquad \text{for any } \nu=0,1,2,...$$

Theorem 3.3.7. *If the family* $\{\varphi_\nu:\nu=0,1,2,...\}$ *is complete in* $L^2(E,\mu)$ (that is, the only element $w\in L^2(E,\mu)$, verifying

$$\int_E w\,\overline{\varphi_\nu}\;d\mu\ \text{ for all } \nu=0,1,2,...,$$

is the zero function), *then*

(a). $T_u^{(\mu)}(g) = \int_E g\, u\, d\mu \qquad (g \in L^2(E,\mu))$;

(b). *each generalized Padé-type approximant* $T_u(g_m(x,z))$ *to* $u \in C^\infty(E)$ *has the integral representation*

$$T_u^{(\mu)}(g_m(x,z)) = \int_E u(x)\, D_m(x,z)\, d\mu(x),$$

where $D_m(x,z)$ *is the kernel*

$$\sum_{k=0}^{m} K_E^{(2)}(\pi_{m,k}) \sum_{j=0}^{m} \frac{\overline{\varphi_j(x)}}{\varphi_j(\pi_{m,k})}.$$

The *Proof* of this *Theorem* is exactly similar to that of *Theorem 3.2.18*. Since

$$K_E(\cdot,\pi_{m,k}) = \overline{K_E(\pi_{m,k},\cdot)} \in L^2(E,\mu),$$

we also have

$$\int_E g(x)\, D_m(x,\cdot)\, d\mu(x) \in L^2(E,\mu)$$

for any $g \in L^2(E,\mu)$. From the *Closed Graph Theorem*, it follows that the integral operator

$$S_\mu^{(m)} : L^2(E,\mu) \to L^2(E,\mu) :: g(\cdot) \mapsto \int_E g(x)\, D_m(x,\cdot)\, d\mu(x)$$

is continuous. Further, by *Fubini's Theorem*, its adjoint operator is given by

$$S_\mu^{(m)*} : L^2(E,\mu) \to L^2(E,\mu) : g(\cdot) \mapsto S_\mu^{(m)}*(g) = \int_E g(x)\, \overline{D_m(\cdot,x)}\, d\mu(x).$$

Definition 3.3.8. *The restriction of* $S_\mu^{(m)}$ *to* $C^\infty(E)$ *is called a generalized Padé-type operator* *for* $C^{\infty}(E)$. *We denote*

$$T^{(\mu)}(g_m(x,\cdot)) := S_{\mu}^{(m)} / C^{\infty}(E).$$

The continuity property for this operator

$$T^{(\mu)}(g_m(x,\cdot)) : C^{\infty}(E) \to C^{\infty}(E) : u(\cdot) \mapsto T_u^{(\mu)}(g_m(x,\cdot)) = \int_E u(x,\cdot)\, D_m(x,\cdot)\, d\mu(x)$$

is a useful tool for the study of convergence and in this connection we give the:

Theorem 3.3.9. *If the sequence*

$$\{u_{\nu} \in C^{\infty}(E) : \nu = 0,1,2,\ldots\}$$

converges to $u \in C^{\infty}(E)$ *with respect to the* $L^2 -$ *norm of* $L^2(E,\mu)$, *then*

$$\lim_{\nu\to\infty} T_{u_{\nu}}^{(\mu)}(g_m(x,\cdot)) = T_u^{(\mu)}(g_m(x,\cdot))$$

in $L^2(E,\mu)$.

Corollary 3.3.10. *If the series of functions*

$$\sum_{\nu=0}^{\infty} \alpha_{\nu}\, u_{\nu}(z) \quad (\alpha_{\nu} \in \mathbb{C}, u_{\nu} \in C^{\infty}(E))$$

converges to $u \in C^{\infty}(E)$ *with respect to the* $L^2 -$ *norm of* $L^2(E,\mu)$, *then*

$$T_u^{(\mu)}(g_m(x,\cdot)) = \sum_{\nu=0}^{\infty} \alpha_{\nu}\, T_{u_{\nu}}^{(\mu)}(g_m(x,\cdot))$$

in $L^2(E,\mu)^2$.

Until now, we have supposed that the compact set E satisfies Markov's inequality (M_2)

with respect to some positive measure μ on E. We will now turn to the case where E fulfills Markov's property (M_∞). As it is pointed out in *Paragraph 3.3.1*(*Theorem 3.3.3*), if $E \subset\subset \mathrm{R}^n$ verifies (M_∞), then there is a Hilbert space $(H, \langle \cdot / \cdot \rangle_H)$ and an orthonormal system $\{\psi_j : |\deg \psi_j| \leq |\deg \psi_{j+1}|, j = 0,1,2,...\}$ in H, such that the injections

$$C^\infty(E) \to (H, \langle \cdot / \cdot \rangle_H) \text{ and } (H, \langle \cdot / \cdot \rangle_H) \to C(E)$$

are continuous and each function $u \in C^\infty(E)$ has the Fourier expansion

$$u(z) = \sum_{j=0}^{\infty} c_j(u) \psi_j(z),$$

where $c_j(u) = \langle u / \psi_j \rangle_H$ and where the series converges uniformly on E.

As for the $(\mathrm{M}_2)-$ case, we shall assume that

$$\{\psi_j : j = 0,1,2,...\}$$

is a self - summable family in $(H, \langle \cdot / \cdot \rangle_H)$, i.e. *for any* $z \in E$, *the sequence*

$$\{\psi_j(z)\overline{\psi_j} : j = 0,1,2,...\}$$

is summable in $(H, \langle \cdot / \cdot \rangle_H)$, in the sense that for every $z \in E$ and every $\varepsilon > 0$ there exists a finite set $J_0 = J_0(z, \varepsilon)$ of indices with

$$\left\| \sum_{j \in J} \psi_j(z)\overline{\psi_j} \right\|_H := \langle \sum_{j \in J} \psi_j(z)\overline{\psi_j} / \sum_{j \in J} \psi_j(z)\overline{\psi_j} \rangle_H^{1/2} < \varepsilon,$$

whenever J is a finite set of indices disjoint from J_0. This summability condition implies that for each $z \in E$ fixed, the function

$$K_E^{(\infty)}(z, \cdot) : E \to \mathbb{C} : x \mapsto K_E^{(\infty)}(z, x) := \sum_{j=0}^{\infty} \psi_j(z)\overline{\psi_j(x)}$$

belongs to H. Note that, by construction $C^\infty(E) \subset H \subset C(E)$ and hence, for any $z \in E$ fixed,

$$K_E^{(\infty)}(z,\cdot) \in C^\infty(E);$$

further, by the continuity of the injective map $(H, \langle \cdot / \cdot \rangle_H) \to C^\infty(E)$, the series

$$\sum_{j=0}^{\infty} \psi_j(z)\overline{\psi_j(\cdot)}$$

converges uniformly on E to $K_E^{(\infty)}(z,\cdot)$.

Let now u be a $C^\infty -$continuous complex-valued function defined on the compact set E satisfying (M_∞). As for the $(\mathrm{M}_2)-$case, we define the linear functional:

$$T_u : \overline{\Psi(\mathbb{C}^n)} \to \mathbb{C} : \overline{\psi_j(x)} \mapsto T_u\left(\overline{\psi_j(x)}\right) := c_j(u),$$

where $\overline{\Psi(\mathbb{C}^n)}$ is the complex subspace of H, which is generated by all finite combinations of $\overline{\psi_j}$'s. If

$$p(x) = \sum_{\nu=0}^{m} \beta_\nu \overline{\psi_\nu(x)} \in \overline{\Psi(\mathbb{C}^n)},$$

then

$$\begin{aligned} \left|T_u(p(x))\right| &= \left|T_u\left(\sum_{\nu=0}^{m} \beta_\nu \overline{\psi_\nu(x)}\right)\right| = \left|\sum_{\nu=0}^{m} \beta_\nu T_u\left(\overline{\psi_\nu(x)}\right)\right| \\ &= \left|\sum_{\nu=0}^{m} \beta_\nu c_\nu(u)\right| = \left|\sum_{\nu=0}^{m} \beta_\nu \langle u / \psi_\nu \rangle_H\right| \\ &= \left|\langle u / \overline{\sum_{\nu=0}^{m} \beta_\nu \overline{\psi_\nu}} \rangle_H\right| = \left|\langle u / \overline{p} \rangle_H\right|. \end{aligned}$$

From *Schwarz's Inequality*, it follows that

$$T_u(p) \leq \|u\|_H \ \|p\|_H,$$

and, by the *Hahn* $-$ *Banach Theorem*, T_u extends to a continuous linear functional on H. For each $z \in E$ fixed, one can therefore define the number

$$T_u\left(K_E^{(\infty)}(z,x)\right),$$

where T_u acts on the variable $x \in E$. Moreover, by continuity, we have

$$u(z) = \sum_{j=0}^{\infty} c_j(u)\psi_j(z) = \sum_{j=0}^{\infty} T_u\left(\overline{\psi_j(x)}\psi_j(z)\right)$$
$$= T_u\left(\sum_{j=0}^{\infty}\psi_j(z)\overline{\psi_j(x)}\right) = T_u\left(K_E^{(\infty)}(z,x)\right).$$

Thus, *computing* $u(z)$ *for a fixed value of* z *is nothing else than computing*

$$T_u\left(K_E^{(\infty)}(z,x)\right).$$

If only a few Fourier coefficients $c_j(u)$ of u are known or if the Fourier series expansion of u (with respect to the family $\{\psi_j : j = 0,1,2,...\}$) converges too slowly, then the function $K_E^{(\infty)}(z,x)$has to be replaced by a simpler expression.

To do so, for any $m = 0,1,2,...$ consider the $(m+1)-$dimensional complex vector space $\overline{\Upsilon}_{m+1}$, generated by the Tchebycheff system

$\{\overline{\psi_0},\overline{\psi_1},...\overline{\psi_m},\}$

and assume that $\overline{\Upsilon}_{m+1}$ satisfies the Haar condition into a finite set of pair-wise distinct points

$$\mathrm{M}_{m+1} = \{\pi_{m,0},\pi_{m,1},...\pi_{m,m}\} \subset E - \left(\bigcup_{0\le j\le m} Ker\overline{\psi_j}\right),$$

that is

$$\det\left[\overline{\psi_j(\pi_{m,k})}\right]_{k,j} := \begin{vmatrix} \overline{\psi_0(\pi_{m,0})} & \overline{\psi_1(\pi_{m,0})} & \cdots & \overline{\psi_m(\pi_{m,0})} \\ \overline{\psi_0(\pi_{m,1})} & \overline{\psi_1(\pi_{m,1})} & \cdots & \overline{\psi_m(\pi_{m,1})} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \overline{\psi_0(\pi_{m,m})} & \overline{\psi_1(\pi_{m,m})} & \cdots & \overline{\psi_m(\pi_{m,m})} \end{vmatrix} \neq 0 .$$

This is equivalent to the fact that for any $z \in E$ there is a unique element

$$g_m(x,z) = \sum_{j=0}^{m} \sigma_j^{(m)}(z)\overline{\psi_j(x)},$$

in $\overline{\mathrm{Y}}_{m+1}$ fulfilling

$$g_m(\pi_{m,k}, z) = \sum_{j=0}^{m} \sigma_j^{(m)}(z)\overline{\psi_j(\pi_{m,k})} = K_E^{(\infty)}(z, \pi_{m,k}), \text{ for } k \le m.$$

Evidently, for $j = 0,1,2,\dots,m$, we have

$$(P_j) \qquad \sigma_j^{(m)}(z) = \sum_{k=0}^{m} \frac{K_E^{(\infty)}(z, \pi_{m,k})}{\psi_j(\pi_{m,k})}.$$

Definition 3.3.11. *Any continuous function $(GPTA/m)_u(z)$, defined by*

$$T_u(g_m(x,\cdot)) : E \to \mathbb{C} : z \mapsto T_u(g_m(x,z)) = \sum_{j=0}^{m} c_j(u)\sigma_j^{(m)}(z),$$

is called a generalized Padé-type approximant to $u \in C^{\infty}(E)$ with generating system M_{m+1}.

If, for each $\nu = 0,1,2,\dots,m$,

$$\sum_{\substack{j=0 \\ (j \ne \nu)}}^{m} c_j(u) \sum_{k=0}^{m} \frac{\overline{\psi_\nu(\pi_{m,k})}}{\overline{\psi_j(\pi_{m,k})}} = 0,$$

then $T_u(g_m(x,\cdot))$ is said to be a Padé-type approximant to u. It is denoted by

$$(PTA/m)_u(z).$$

Obviously, the computation of a generalized Padé-type approximant $T_u(g_m(x,z))$ requires only the knowledge of the Fourier coefficients

$$c_0(u), c_1(u), \dots, c_m(u)$$

and of the functions

$$\sigma_0^{(m)}(z), \sigma_1^{(m)}(u), \ldots, \sigma_m^{(m)}(z)$$

resulting from $(P_0), (P_1), \ldots, (P_m)$, respectively.

Theorem 3.3.12. *If*

$$\sum_{\nu=0}^{\infty} \beta_\nu^{(m,u)} \psi_\nu(z)$$

is the Fourier expansion of a Padé-type approximant $(PTA/m)_u(z) = T_u(g_m(x,z))$ *to*

$$u(z) = \sum_{\nu=0}^{\infty} c_\nu(u)\, \psi_\nu(z) \in C^\infty(E),$$

then for any $\nu = 0,1,2,\ldots,m$, *there holds*

$$\beta_\nu^{(m,u)} = c_\nu(u).$$

Proof. Since $K_E^{(\infty)}(\cdot, \pi_{m,k}) \in H$ for any $k \leq m$, each function

$$\sigma_j^{(m)}(z) = \sum_{k=0}^{m} \frac{K_E^{(\infty)}(z, \pi_{m,k})}{\psi_j(\pi_{m,k})}$$

belongs to the Hilbert space H, for every $j = 0,1,2,\ldots,m$. It follows that there are Fourier coefficients

$$t_\nu^{(j,m)} = \langle \sigma_j^{(m)} / \psi_\nu \rangle_H$$

such that

$$\sigma_j^{(m)}(z) = \sum_{\nu=0}^{m} t_\nu^{(j,m)} \psi_\nu(z)$$

uniformly on E. We can therefore write

$$T_u\left(g_m(x,z)\right)=\sum_{j=0}^{m} c_j(u)\,\sigma_j^{(m)}(z)$$

$$=\sum_{j=0}^{m} c_j(u)\sum_{\nu=0}^{\infty} t_\nu^{(j,m)}\,\psi_\nu(z)$$

$$=\sum_{\nu=0}^{\infty}\left(\sum_{j=0}^{m} t_\nu^{(j,m)}\,c_j(u)\right)\psi_\nu(z).$$

This implies that

$$\beta_\nu^{(m,u)}=\sum_{j=0}^{m} t_\nu^{(j,m)} c_j(u)$$

or alternatively that

$$\beta_\nu^{(m,,u)}=\sum_{j=0}^{m}\langle\sigma_\nu^{(m)}/\psi_\nu\rangle_H\,c_j(u)$$

$$=\sum_{j=0}^{m}\langle\sum_{k=0}^{m}\frac{K_E^{(\infty)}(\cdot,\pi_{m,k})}{\psi_j(\pi_{m,k})}/\psi_\nu\rangle_H\,c_j(u)$$

$$=\sum_{j=0}^{m}\sum_{k=0}^{m}\frac{1}{\psi_j(\pi_{m,k})}\langle K_E^{(\infty)}(\cdot,\pi_{m,k})/\psi_\nu\rangle_H\,c_j(u).$$

Now, since, for any $k\le m$,

$$K_E^{(\infty)}(\cdot,\pi_{m,k})=\overline{K_E^{(\infty)}(\pi_{m,k},\cdot)}$$

(or since the series

$$\sum_{j=0}^{\infty}\psi_j(\cdot)\overline{\psi_j(\pi_{m,k})}$$

converges uniformly on E), observe that the family

$$\left\{\psi_j(\cdot)\cdot\overline{\psi_j(\pi_{m,k})}:\ j=0,1,2...,\right\}$$

is summable in H with sum

$$K_E^{(\infty)}(\cdot,\pi_{m,k})\in H\subset C(E).$$

By orthonormality, we obtain

$$\langle K_E^{(\infty)}(\cdot,\pi_{m,k})/\psi_\nu\rangle_H=\langle\sum_{j=0}^{\infty}\psi_j(\cdot)\,\overline{\psi_j(\pi_{m,\kappa})}/\psi_\nu\rangle_H=$$

$$=\sum_{j=0}^{\infty}\overline{\psi_j(\pi_{m,k})}\langle\psi_j/\psi_\nu\rangle_H$$

$$=\overline{\psi_\nu(\pi_{m,k})}\quad(k\le m).$$

Hence, from the definition of the Padé-type approximant $T_u(g_m(x,z))$, it follows that

$$\beta_\nu^{(m,u)}=\sum_{j=0}^{m}\sum_{k=0}^{m}\frac{\overline{\psi_\nu(\pi_{m,k})}}{\overline{\psi_j(\pi_{m,k})}}c_j(u)=c_\nu(u)+\sum_{\substack{j=0\\(j\neq\nu)}}^{m}c_j(u)\sum_{k=0}^{m}\frac{\overline{\psi_\nu(\pi_{m,k})}}{\overline{\psi_j(\pi_{m,k})}}=c_\nu(u),$$

for any $\nu=0,1,2,...,m$. This ends the *Proof.*

A criterion for the efficiency of generalized Padé-type approximants to continuous functions on E is , of course, their convergence behavior. This problem, connected with the best choices of the orthonormal system

$$\{\psi_j : j=0,1,2,...\}$$

and of the generating system

$$\mathrm{M}_{m+1}=\{\pi_{m,0},\pi_{m,1},...\pi_{m,m}\},$$

will be discussed below in *Theorem 3.3.14.*

First, let us study the errors. It is easily verified that

Theorem 3.3.13 (a). *The error of a generalized Padé-type approximation to*

$$u(z) = \sum_{\nu=0}^{\infty} c_\nu(u)\, \psi_\nu(z) \in C^\infty(E)$$

equals

$$T_u\left(g_m(x,z)\right) - u(z) = \sum_{\nu=0}^{\infty}\left[\sum_{j=0}^{m} c_j(u) \sum_{k=0}^{m} \frac{\overline{\psi_\nu(\pi_{m,k})}}{\overline{\psi_j(\pi_{m,k})}} - c_\nu(u)\right]\psi_\nu(z), \quad z \in E.$$

(b). *The error of a Padé-type approximation to*

$$u(z) = \sum_{\nu=0}^{\infty} c_\nu(u)\, \psi_\nu(z) \in C^\infty(E)$$

is

$$T_u\left(g_m(x,z)\right) - u(z) = \sum_{\nu=m+1}^{\infty}\left[\sum_{j=0}^{m} c_j(u) \sum_{k=0}^{m} \frac{\overline{\psi_\nu(\pi_{m,k})}}{\overline{\psi_j(\pi_{m,k})}} - c_\nu(u)\right]\psi_\nu(z), \quad z \in E.$$

We can immediately obtain an answer to the convergence problem of a generalized Padé-type approximation sequence.

Theorem 3.3.14. *Let* E *be a compact subset of* R^n *satisfying Markov's inequality* (M_∞) *and let* $u \in C^\infty(E)$.

Consider the intermediate Hilbert space $(H, \langle \cdot / \cdot \rangle_H)$, *for which the natural injections* $C^\infty(E) \to H \to C(E)$ *are continuous. Suppose*

$$\{\psi_j : j = 0,1,2,\ldots\}$$

is a self-summable family consisting of orthonormal polynomials in H, *such that*

$$\left|\deg \psi_j\right| \le \left|\deg \psi_{j+1}\right| \quad (j = 0,1,2,...)$$

and assume that the function

$$K_E^{(\infty)}(\cdot,\cdot): E \to \mathbb{C}: z \mapsto K_E^{(\infty)}(z,z) := \sum_{\nu=0}^{\infty} \left|\psi_\nu(z)\right|^2$$

is continuous on E. *Let also*

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m\ge 0, 0\le k\le m}$$

be an infinite triangular matrix, with elements $\pi_{m,k} \in \Omega$, *such that for any* $m \ge 0$

$$\pi_{m,k} \ne \pi_{m,k'} \ (if\ k \ne k'),\ \pi_{m,k} \notin \left(\bigcup_{0\le j\le m} Ker\overline{\psi_j}\right) \ (k \le m) \ and\ \det\left[\overline{\psi_j(\pi_{m,k})}\right]_{k,j} \ne 0.$$

If

$$\lim_{m\to\infty}\left\{\sum_{\nu=0}^{\infty}\left\|\sum_{j=0}^{m}\psi_j\left(\sum_{k=0}^{m}\frac{\psi_\nu(\pi_{m,k})}{\psi_j(\pi_{m,k})}\right) - \psi_\nu\right\|_H^2\right\} = 0,$$

then, the corresponding generalized Padé-type approximation sequence to $u(z)$

$$\left\{T_u\left(g_m(x,z)\right): m = 0,1,2,...\right\}$$

converges to $u(z)$ *uniformly on* E.

Proof. By *Theorem 3.3.9* and by *Cauchy – Schwarz's Inequality*, we have

$$\sup_{z\in E}\left|T_u\left(g_m(x,z)\right) - u(z)\right| = \sup_{z\in E}\left|\sum_{\nu=0}^{\infty}\langle u / \left(\sum_{j=0}^{m}\psi_j \sum_{k=0}^{m}\frac{\psi_\nu(\pi_{m,k})}{\psi_j(\pi_{m,k})} - \psi_\nu\right)\rangle_H \ \psi_\nu(z)\right|$$

$$\le \sup_{z\in E}\left\{\sum_{\nu=0}^{\infty}\left|\psi_\nu(z)\right|^2\right\}^{\frac{1}{2}} \left\| u \right\|_H \left\{\sum_{\nu=0}^{\infty}\left\|\sum_{j=0}^{m}\frac{\psi_\nu(\pi_{m,k})}{\psi_j(\pi_{m,k})} - \psi_\nu\right\|_H^2\right\}^{\frac{1}{2}}.$$

Since, by the continuity of $K_E^{(\infty)}(z,z)$,

$$\sup_{z\in E}\left\{\sum_{\nu=0}^{\infty}\left|\psi_\nu(z)\right|^2\right\}^{\frac{1}{2}}=\sup_{z\in E}\left\{\sum_{\nu=0}^{\infty}\psi_\nu(z)\overline{\psi_\nu(z)}\right\}^{\frac{1}{2}}=\sup_{z\in E}\left\{K_E^{(2)}(z,z)\right\}^{\frac{1}{2}}=:\sigma(E),$$

and $\sigma(E)<\infty$, the *Proof* is complete.

3.4 The Hilbert Space Case

3.4.1. Generalized Padé-type Approximation in Functional Hilbert Spaces

It is readily seen that the methods of *Paragraph 3.3.2* can be extended into a functional Hilbert space.

Let $H\neq\{0\}$ be any Hilbert space, consisting of functions defined into an arbitrary topological space X and with values into the extended complex plane $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. Let $\langle\cdot/\cdot\rangle_H$ be the inner product of H and let $\|\cdot\|_H$ be the corresponding norm. Assume that H is enough large, so that if $f\in H$ then $\overline{f}\in H$.

Suppose

$$\mathrm{N}=\{e_j : j=1,2,...\}$$

is a countable *complete* orthonormal set in H. The condition that N is complete means that the only vector orthogonal to every e_j is the zero vector. Then, each $u\in H$ has the Fourier expansion

$$u(x)=\sum_{j=0}^{\infty}\langle u/e_j\rangle_H\, e_j(z)\qquad (z\in \mathrm{X}).$$

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Further, assume that N is a *self-summable* family in H, in the sense that for any $z \in \mathrm{X}$ the sequence

$$\left\{ e_j(z)\overline{e_j} : j = 0,1,2,\ldots \right\}$$

is summable in H ([74]). This summability condition guarantees that, for each $z \in \mathrm{X}$ fixed, the function

$$K_{\mathrm{X}}(z,\cdot) : \mathrm{X} \to \mathbb{C} : x \mapsto K_{\mathrm{X}}(z,x) = \sum_{j=0}^{\infty} e_j(z)\overline{e_j(x)}$$

belongs to H.

First, consider any element $u \in H$ and introduce the linear functional $T_u : \overline{E} \to \mathbb{C}$ defined by

$$T_u\left(\overline{e_j(x)}\right) := \langle u / e_j \rangle_H ,$$

where $\overline{E}$ is the complex vector subspace of H which is generated by all finite complex combinations of $\overline{e_j}$'s. If

$$p(x) = \sum_{\nu=0}^{m} \beta_\nu \overline{e_\nu(x)} \in \overline{E},$$

then

$$\left|T_u(p(x))\right| = \left| T_u\left(\sum_{\nu=0}^{m} \beta_\nu \overline{e_\nu(x)} \right) \right| = \left| \sum_{\nu=0}^{m} \beta_\nu \langle u / e_\nu \rangle_H \right|$$
$$= \left| \langle u / \overline{\sum_{\nu=0}^{m} \beta_\nu \overline{e_\nu}} \rangle_H \right| = \left| \langle u / \overline{p} \rangle_H \right|$$

and by *Schwarz' Inequality*,

$$\left|T_u(p(x))\right| \le \|u\|_H \, \|p\|_H .$$

It follows, from the *Hahn – Banach Theorem*, that T_u extends to a linear continuous functional on H. For every $z \in \mathrm{X}$ fixed, one can therefore define the number

$$T_u\left(K_{\mathrm{X}}(z,x)\right),$$

where T_u acts on the variable $x \in \mathrm{X}$. Moreover, by continuity, there holds

$$u(z)=\sum_{j=0}^{\infty}\langle u/e_j\rangle_H\, e_j(z)=\sum_{j=0}^{\infty}T_u\left(\overline{e_j(x)}\right)e_j(z)=T_u\left(\sum_{j=0}^{\infty}e_j(z)\overline{e_j(x)}\right)=T_u\left(K_{\mathrm{X}}(z,x)\right).$$

Thus, computing $u(z)$ for a fixed value of $z \in \mathrm{X}$ is nothing else than computing

$$T_u\left(K_{\mathrm{X}}(z,x)\right).$$

It arises in practice that only a few Fourier coefficients $<u/e_j>_H$ of u are known or that the Fourier expansion of u with respect to the basis $\{e_j : j=1,2,...\}$ converges too slowly. Thus, the function $K_{\mathrm{X}}(z,x)$ has to be replaced by a simpler expression. Our method will therefore follow the ideas of *Paragraphs 3.2.3* and *3.3.2*.

For any $m=0,1,2,...$, let us consider the $(m+1)$ – dimensional complex vector space $\overline{E}_{m+1}$, spanned by the Tchebycheff system

$$\{e_0, e_1, ..., e_m\}$$

and assume that $\overline{E}_{m+1}$ satisfies Haar's condition in a finite set of pair-wise distinct points.

$$\mathrm{M}_{m+1}=\{\pi_{m,0},\pi_{m,1},...,\pi_{m,m}\}\subset X-\left(\bigcup_{0\le j\le m}Ker\overline{e_j}\right)$$

($Ker\overline{e_j}$ is the kernel of $\overline{e_j}$), that is

$$\det\left[\overline{e_j(\pi_{m,k})}\right]_{k,j}=\begin{vmatrix}\overline{e_0(\pi_{m,0})} & \overline{e_1(\pi_{m,0})}\ldots\overline{e_m(\pi_{m,0})}\\ \overline{e_0(\pi_{m,1})} & \overline{e_1(\pi_{m,1})}\ldots\overline{e_m(\pi_{m,1})}\\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\\ \overline{e_0(\pi_{m,m})} & \overline{e_1(\pi_{m,m})}\ldots\overline{e_m(\pi_{m,m})}\end{vmatrix}\neq 0.$$

By *Theorem 3.2.5*, this is equivalent to the fact that for any $z \in \mathrm{X}$ there exists a unique element in $\overline{E}_{m+1}$:

$$G_m(x,z) = \sum_{j=0}^{m} \sigma_j^{(m)}(z)\overline{e_j(x)},$$

fulfilling

$$G_m(x,z) = K_{\mathrm{X}}(z, \pi_{m,k}) \qquad \text{for any } k \le m \,.$$

Definition 3.4.1. *Any function* $(GPTA/m)_u(z)$, *defined by*

$$T_m(G_m(x,\cdot)): X \to \overline{\mathbb{C}} : z \mapsto T_u(G_m(x,z)) = \sum_{j=0}^{m} \langle u/e_j \rangle_H \, \sigma_j^{(m)}(z),$$

is an element of H *and is called a generalized Padé-type approximant of* $u \in H$ *with generating system* M_{m+1}.

If

$$\sum_{\substack{j=0 \\ (j \neq \nu)}}^{m} \langle u/e_j \rangle_H \sum_{k=0}^{m} \frac{\overline{e_\nu(\pi_{m,k})}}{e_j(\pi_{m,k})} = 0 \qquad \text{for any } \nu = 0,1,2,\ldots,m,$$

then $T_u(G_m(x,\cdot))$ *is said to be a Padé-type approximant to* u ; *it is denoted by* $(PTA/m)_u(z)$.

Obviously, the computation of a generalized Padé-type approximant $(GPTA/m)_u(z) = T_u(G_m(x,\cdot))$ to $u \in H$ requires only the knowledge of the Fourier coefficients

$$< u/e_0 >_H, < u/e_1 >_H, \ldots, < u/e_m >_H$$

and the functional quantities

$$\sigma_0^{(m)}(\cdot), \sigma_1^{(m)}(\cdot), \ldots, \sigma_m^{(m)}(\cdot)$$

resulting from the equations

$$(H_j) \qquad \sigma_j^{(m)}(\cdot)=\sum_{k=0}^{m}\frac{K_X(\cdot,\pi_{m,k})}{\overline{e_j(\pi_{m,k})}}, \qquad j=0,1,2,\ldots,m.$$

The following result justifies the notation "*Padé-type approximant to* u":

Theorem 3.4.2. *If*

$$(PTA/m)_u(z)=T_u(G_m(x,\cdot))$$

is a Padé-type approximant to $u\in H$ *, then there holds*

$$<T_u(G_m(x,\cdot))/e_\nu>_H=<u/e_\nu>_H, \qquad \textit{for any } \nu=0,1,\ldots,m.$$

Proof. Since $K_X(\cdot,\pi_{m,\kappa})\in H \quad (k\le m)$, it is immediately seen that

$$\sigma_j^{(m)}(\cdot)=\sum_{k=0}^{m}\frac{K_X(\cdot,\pi_{m,k})}{\overline{e_j(\pi_{m,k})}}\in H, \qquad j=0,1,2,\ldots,m,$$

and consequently

$$\sigma_j^{(m)}(\cdot)=\sum_{\nu=0}^{\infty}\langle\sigma_j^{(m)}/e_\nu\rangle_H\, e_\nu(\cdot) \qquad (j=0,1,2,\ldots,m).$$

One can therefore write

$$\begin{aligned}T_u(G_m(x,\cdot))&=\sum_{j=0}^{m}\langle u/e_j\rangle_H\,\sigma_j^{(m)}(\cdot)\\&=\sum_{j=0}^{\infty}\langle u/e_j\rangle_H\sum_{\nu=0}^{\infty}\langle\sigma_j^{(m)}/e_\nu\rangle_H\,e_\nu(\cdot)\\&=\sum_{\nu=0}^{\infty}\left[\sum_{j=0}^{m}\langle\sigma_j^{(m)}/e_\nu\rangle_H\,\langle u/e_j\rangle_H\right]e_\nu(\cdot),\end{aligned}$$

which implies that, for any $\nu=0,1,2,\ldots,$ there holds

$$
\begin{aligned}
< T_u\left(G_m(x,\cdot)\right)/e_\nu >_H &= \sum_{j=0}^{m} \langle \sigma_j^{(m)} / e_\nu \rangle_H \langle u / e_j \rangle_H \\
&= \sum_{j=0}^{m} \langle \sum_{k=0}^{m} \frac{K_X\left(\cdot, \pi_{m,k}\right)}{e_j\left(\pi_{m,k}\right)} / e_\nu \rangle_H \langle u / e_j \rangle_H \\
&= \sum_{j=0}^{m} \sum_{k=0}^{m} \frac{1}{\overline{e_j\left(\pi_{m,k}\right)}} \langle K_X\left(\cdot, \pi_{m,k}\right) / e_\nu \rangle_H \langle u / e_j \rangle_H \ .
\end{aligned}
$$

Observe that, for any $k \leq m$, the family

$$
\left\{ e_j(\cdot)\overline{e_j\left(\pi_{m,k}\right)} : j = 0,1,2,\ldots \right\}
$$

is summable in H with sum $K_X\left(\cdot, \pi_{m,k}\right) \in H$ (because of the symmetry property

$$
K_X\left(\cdot, \pi_{m,k}\right) = \overline{K_X\left(\pi_{m,k}, \cdot\right)}).
$$

By orthonormality, we obtain

$$
\begin{aligned}
\langle K_X\left(\cdot, \pi_{m,k}\right) / e_\nu \rangle &= \langle \sum_{j=0}^{\infty} e_j(\cdot)\overline{e_j\left(\pi_{m,k}\right)} / e_\nu \rangle_H \\
&= \sum_{j=0}^{\infty} \overline{e_j\left(\pi_{m,k}\right)} \langle e_j / e_\nu \rangle_H \\
&= \overline{e_\nu\left(\pi_{m,k}\right)}
\end{aligned}
$$

$(k \leq m)$. From the definition of the Padé-type approximant $T_u\left(G_m(x,\cdot)\right)$, it follows that

$$
\begin{aligned}
< T_u\left(G_m(x,\cdot)\right)/e_\nu >_H &= \sum_{j=0}^{m} \sum_{k=0}^{m} \frac{\overline{e_\nu\left(\pi_{m,k}\right)}}{\overline{e_j\left(\pi_{m,k}\right)}} \langle u / e_j \rangle_H \\
&= \langle u / e_\nu \rangle_H + \sum_{\substack{j=0 \\ (j \neq \nu)}}^{m} \langle u / e_j \rangle_H \sum_{k=0}^{m} \frac{\overline{e_\nu\left(\pi_{m,k}\right)}}{\overline{e_j\left(\pi_{m,k}\right)}} \\
&= \langle u / e_\nu \rangle_H \ ,
\end{aligned}
$$

for any $\nu = 0,1,2,\ldots,m$, which completes the *Proof.*

Theorem 3.4.3. (a). *The error of a generalized Padé-type approximation is*

$$(GPTA/m)_u(\cdot)-u(\cdot)=\sum_{\nu=0}^{\infty}\left[\sum_{\substack{j=0\\(j\neq\nu)}}^{m}\langle u/e_j\rangle_H\sum_{k=0}^{m}\frac{\overline{e_\nu(\pi_{m,k})}}{\overline{e_j(\pi_{m,k})}}\right]e_\nu(\cdot).$$

(b). *The error of a Padé-type approximation equals*

$$(PTA/m)_u(\cdot)-u(\cdot)=\sum_{\nu=m+1}^{\infty}\left[\sum_{\substack{j=0\\(j\neq\nu)}}^{m}\langle u/e_j\rangle_H\sum_{k=0}^{m}\frac{\overline{e_\nu(\pi_{m,k})}}{\overline{e_j(\pi_{m,k})}}\right]e_\nu(\cdot).$$

Proof. (a). If $T_u(G_m(x,\cdot))$ is a generalized Padé-type approximant to $u(\cdot)\in H$, then

$$T_u(G_m(x,\cdot))-u(\cdot)=\sum_{\nu=0}^{\infty}[\langle T_u(G_m(x,\cdot))/e_\nu\rangle_H-\langle u/e_\nu\rangle_H]e_\nu(\cdot)$$

$$=\sum_{\nu=0}^{\infty}\left[\sum_{j=0}^{m}\langle\sigma_j^{(m)}/e_\nu\rangle_H\langle u/e_j\rangle_h-\langle u/e_\nu\rangle_H\right]e_\nu(\cdot)$$

$$=\sum_{\nu=0}^{\infty}\left[\sum_{j=0}^{m}\sum_{k=0}^{m}\frac{1}{e_j(\pi_{m,k})}\langle K_X(\cdot,\pi_{m,k})/e_\nu\rangle_H\langle u/e_j\rangle_H-\langle u/e_\nu\rangle_H\right]e_\nu(\cdot)$$

$$=\sum_{\nu=0}^{\infty}\left[\sum_{j=0}^{m}\sum_{k=0}^{m}\frac{\overline{e_\nu(\pi_{m,k})}}{\overline{e_j(\pi_{m,k})}}\langle u/e_j\rangle_H-\langle u/e_\nu\rangle_H\right]e_\nu(\cdot)$$

$$=\sum_{\nu=0}^{\infty}\left[\sum_{\substack{j=0\\(j\neq\nu)}}^{m}\langle u/e_j\rangle_H\sum_{k=0}^{m}\frac{\overline{e_\nu(\pi_{m,k})}}{\overline{e_j(\pi_{m,k})}}\right]e_\nu(\cdot).$$

(b). If $T_u(G_m(x,\cdot))$ is a Padé-type approximant to $u(\cdot)\in H$, then repetition of the *Proof* of *(a)* shows that

$$T_u(G_m(x,\cdot))-u(\cdot)=\sum_{\nu=m+1}^{\infty}\left[\sum_{\substack{j=0\\(j\neq\nu)}}^{m}\langle u/e_j\rangle_H\sum_{k=0}^{m}\frac{\overline{e_\nu(\pi_{m,k})}}{\overline{e_j(\pi_{m,k})}}\right]e_\nu(\cdot).$$

By *Schwarz's Inequality*, we directly get the following:

Corollary 3.4.4. *Let* $u(\cdot) \in H$ *. Let also*

$$M = (\pi_{m,k})_{m \geq 0,\ 0 \leq k \leq m}$$

be an infinite triangular matrix, with elements $\pi_{m,k} \in \Omega$, *such that for any* $m \geq 0$

$$\pi_{m,k} \neq \pi_{m,k'} \ (if \ \ k \neq k'),$$

$$\pi_{m,k} \notin \left(\bigcup_{0 \leq j \leq m} Ker \overline{e_j(\cdot)} \right) \quad (if \ \ k \leq m)$$

and

$$\det \left[\overline{e_j(\pi_{m,k})} \right]_{k,j} \neq 0 .$$

Assume that

$$\lim_{m \to \infty} \left\{ \sum_{\nu=0}^{\infty} \left\| \sum_{\substack{j=0 \\ (j \neq \nu)}}^{m} e_j(\cdot) \left[\sum_{k=0}^{m} \frac{e_\nu(\pi_{m,k})}{e_j(\pi_{m,k})} \right] \right\|_H^2 \right\} = 0 .$$

If

$$\sup_{z \in E} \left| K_{\mathrm{X}}(z,z) \right|^{\frac{1}{2}} < \infty$$

for some subset E *of* X, *then the restriction to* E *of the corresponding generalized Padé-type approximation sequence*

$$\left\{ (GPTA/m)_u(z) = T_u(G_m(x,\cdot))/_E : m = 0,1,2,\ldots \right\}$$

converges the restriction $u(\cdot)$ *to* E, *uniformly on* E.

The *Proof* of *Corollary 3.4.4* is similar to that of *Theorem 3.3.14*, but in general the assumption that

$$\sup_{z \in E} \left| K_{X}(z,z) \right|^{\frac{1}{2}} < \infty$$

is strong and constitutes a serious obstacle to obtain global converges answers. We can however obtain particular results, which may lead to interesting and satisfactory applications. The first such result is easy and in a sense automatic.

In the sequel, if $w \in H$, $u \in H$ and if $\langle w / u \rangle_H = 0$, then w is said to be orthogonal to u and the notation $w \perp u$ is used. Obviously, the relation $\perp$ is symmetric. If $E \subset H$ and $S \subset H$, the notation $E \perp S$ means that $w \perp u$ whenever $w \in H$ and $u \in H$. Also, $S^{\perp}$ is the set of all $w \in H$ that are orthogonal to every $u \in S$.

Theorem 3.4.5. *Let $u \in H$. Let also $\pi_{m,0}, \pi_{m,1}, \ldots, \pi_{m,m}$ be a finite set of pair-wise distinct points of X such that*

$$\pi_{m,k} \notin \left(\bigcup_{0 \le j \le m} Ker \overline{e_j(\cdot)} \right) \quad (if \;\; k \le m) \;\; and \;\; \det\left[\overline{e_j(\pi_{m,k})} \right]_{k,j} \neq 0 .$$

If

$$\sum_{\substack{j=0 \\ (j \neq v)}}^{m} e_j(\cdot) \sum_{k=0}^{m} \frac{\overline{e_v(\pi_{m,k})}}{\overline{e_j(\pi_{m,k})}} \perp \;\; u(\cdot) \qquad for \; all \;\; v = 0,1,2,\ldots,$$

then there holds

$$\lim_{i \to \infty} T_u\left(G_{m_i}(x,\cdot) \right) = u(\cdot)$$

(where all limits were taken with respect to the $\|\cdot\|_H$ $-$ norm).

As another almost such automatic result, we also mention the following:

Theorem 3.4.6. *Let* S *be a subset of* H. Let also

$$\mathrm{M} = \left(\pi_{m,k}\right)_{m\geq 0, 0\leq k\leq m}$$

be a infinite triangular matrix, with elements $\pi_{m,k} \in \mathrm{X}$ *such that for any* $m \geq 0$

$$\pi_{m,k} \neq \pi_{m,k'} \ (if \ \ k \neq k'),$$

$$\pi_{m,k} \notin \left(\bigcup_{0\leq j\leq m} Ker\overline{e_j(\cdot)} \right) \quad (if \ \ k \leq m)$$

and

$$\det\left[\overline{e_j\left(\pi_{m,k}\right)}\right]_{k,j} \neq 0 .$$

If

$$[\lim_{m\to\infty} \sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j(\cdot) \sum_{k=0}^{m} \frac{\overline{e_\nu\left(\pi_{m,k}\right)}}{e_j\left(\pi_{m,k}\right)}] \in S^{\perp} \qquad \textit{for every } \nu = 0,1,2,...,$$

then for any $u \in S$ *there exists a subsequence* $\{m_i : i = 0,1,2,...\}$ *of* $\{m \geq 0\}$ *such that*

$$\lim_{i\to\infty} T_u\left(G_{m_i}(x,\cdot)\right) = u(\cdot)$$

(where all limits were taken with respect to the $\|\cdot\|_H$ − norm).

Proof. Fix any $u \in S$. Since the function

$$\Lambda : H \to \mathrm{R} \bigcup \{-\infty\} : f \mapsto \Lambda(f) := -\sum_{\nu=0}^{\infty} \left\| f - \left[\lim_{m\to\infty} \sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j(\cdot) \left(\sum_{k=0}^{m} \frac{\overline{e_\nu\left(\pi_{m,k}\right)}}{e_j\left(\pi_{m,k}\right)} \right) \right] \right\|_H$$

is upper semi-continuous (i.e. Λ is the point-wise limit of the decreasing sequence of continuous functions

$$\Lambda_N : H \to \mathbb{R} : f \mapsto \Lambda_N(f) := -\sum_{\nu=0}^{N} \left\| f - \left[\lim_{m\to\infty} \sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j(\cdot) \left(\sum_{k=0}^{m} \frac{e_\nu(\pi_{m,k})}{e_j(\pi_{m,k})} \right) \right] \right\|_H ,$$

we have

$$\overline{\lim}_{m\to\infty} \left\{ \sum_{\nu=0}^{\infty} \left\| \sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j(\cdot) \sum_{k=0}^{m} \frac{e_\nu(\pi_{m,k})}{e_j(\pi_{m,k})} - \left[\lim_{m\to\infty} \sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j(\cdot) \sum_{k=0}^{m} \frac{e_\nu(\pi_{m,k})}{e_j(\pi_{m,k})} \right] \right\|_H \left\| e_\nu(\cdot) \right\|_H \right\} = 0$$

$$\Rightarrow \overline{\lim}_{m\to\infty} \left\{ \sum_{\nu=0}^{\infty} \left| \langle u / \left[\sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j \sum_{k=0}^{m} \frac{e_\nu(\pi_{m,k})}{e_j(\pi_{m,k})} - \lim_{m\to\infty} \sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j \sum_{k=0}^{m} \frac{e_\nu(\pi_{m,k})}{e_j(\pi_{m,k})} \right] \rangle_H \right| \left\| e_\nu(\cdot) \right\|_H \right\} = 0$$

$$\Leftrightarrow \overline{\lim}_{m\to\infty} \left\{ \sum_{\nu=0}^{\infty} \left\| \langle u / \left[\sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j \sum_{k=0}^{m} \frac{e_\nu(\pi_{m,k})}{e_j(\pi_{m,k})} \right] \rangle_H \, e_\nu(\cdot) \right. \right.$$

$$\left. \left. - \langle u / \left[\lim_{m\to\infty} \sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j \sum_{k=0}^{m} \frac{e_\nu(\pi_{m,k})}{e_j(\pi_{m,k})} \right] \rangle_H \, e_\nu(\cdot) \right\|_H \right\} = 0$$

$$\Rightarrow \overline{\lim}_{m\to\infty} \left\{ \left\| \sum_{\nu=0}^{\infty} \langle u / \left[\sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j \sum_{k=0}^{m} \frac{e_\nu(\pi_{m,k})}{e_j(\pi_{m,k})} \right] \rangle_H \, e_\nu(\cdot) \right. \right.$$

$$\left. \left. - \langle u / \left[\lim_{m\to\infty} \sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j \sum_{k=0}^{m} \frac{e_\nu(\pi_{m,k})}{e_j(\pi_{m,k})} \right] \rangle_H \, e_\nu(\cdot) \right\|_H \right\} = 0$$

$$\Leftrightarrow \overline{\lim}_{i\to\infty}\left\{\sum_{\nu=0}^{\infty}\langle u/\left[\sum_{\substack{j=0\\(j\neq\nu)}}^{m_i} e_j \sum_{k=0}^{m}\frac{e_\nu\left(\pi_{m_i,k}\right)}{e_j\left(\pi_{m_i,k}\right)}\right]\rangle_H\ e_\nu(\cdot)\right\}$$

$$=\sum_{\nu=0}^{\infty}\lim_{m\to\infty}\langle u/\left[\sum_{\substack{j=0\\(j\neq\nu)}}^{m} e_j \sum_{k=0}^{m}\frac{e_\nu\left(\pi_{m,k}\right)}{e_j\left(\pi_{m,k}\right)}\right]\rangle_H\ e_\nu(\cdot),$$

for some subsequence $\{m_i : i=0,1,2,...\}$ of $\{m\geq 0\}$. It follows, from *Riesz's Representation Theorem* and from *Theorem 3.4.3.(a)*, that

$$\lim_{i\to\infty}\left[T_u\left(G_{m_i}(x,\cdot)\right)-u(\cdot)\right] \quad =\sum_{\nu=0}^{\infty}\lim_{m\to\infty}\langle u/\left[\sum_{\substack{j=0\\(j\neq\nu)}}^{m} e_j \sum_{k=0}^{m}\frac{e_\nu\left(\pi_{m,k}\right)}{e_j\left(\pi_{m,k}\right)}\right]\rangle_H\ e_\nu(\cdot).$$

Our hypothesis that

$$[\lim_{m\to\infty}\sum_{\substack{j=0\\(j\neq\nu)}}^{m} e_j(\cdot)\sum_{k=0}^{m}\frac{\overline{e_\nu\left(\pi_{m,k}\right)}}{\overline{e_j\left(\pi_{m,k}\right)}}\ ]\in S^{\perp}$$

now shows that

$$\lim_{i\to\infty}\left[T_u\left(G_{m_i}(x,\cdot)\right)-u(\cdot)\right]=0,$$

which completes the *Proof*.

Our next objective is to propose a representation for generalized Padé-type approximants to elements of H. As it is pointed out, the functional T_u extends continuously and linearly on H, whenever $u\in H$. By the *Riesz Representation Theorem*, there exists a unique element $U\in H$ satisfying

$$T_u(g)=<g/U>_H \text{ for all } g\in H.$$

If, in particular, $g(\zeta)=\overline{e_\nu(\zeta)}$, $(\zeta \in \mathrm{X})$, then

$$T_u(\overline{e_\nu}) = <\overline{e_\nu}/U>_H \quad (\nu = 0,1,2,...).$$

Since, by definition, we also have

$$T_u(\overline{e_\nu}) = <u/e_\nu>_H = \overline{<e_\nu/u>_H} \qquad (\nu = 0,1,2,...).$$

we conclude that

$$<\overline{e_\nu}/U>_H - <\overline{e_\nu}/\overline{u}>_H = \overline{<e_\nu/u>_H} - <\overline{e_\nu}/\overline{u}>_H,$$

or equivalently

$$<\overline{e_\nu}/U-\overline{u}>_H = \overline{<e_\nu/u>_H} - <\overline{e_\nu}/\overline{u}>_H, \qquad \text{for all } \nu = 0,1,2,...$$

Theorem 3.4.7. *Suppose the inner product of any two real-valued functions in H is real. Then*

(a). $T_u(g) = <g/\overline{u}>_H$, *whenever* $g \in H$;

(b). *every generalized Padé-type approximant* $T_u(G_m(x,\cdot))$ *to* $u(\cdot) \in H$, *is written in the following form :*

$$T_u(G_m(x,z)) = <u/\overline{D_m(\cdot,z)}>_H \qquad (z \in \mathrm{X})$$

with

$$D_m(x,z) = \sum_{k=0}^{m} K_{\mathrm{X}}(z,\pi_{m,k}) \sum_{j=0}^{m} \frac{\overline{e_j(x)}}{\overline{e_j(\pi_{m,k})}}.$$

Proof. Let $u = (u^{(1)} + iu^{(2)}) \in H$ and $w = (w^{(1)} + iw^{(2)}) \in H$. It is readily seen that

$$<\overline{w}/\overline{u}>_H = <w^{(1)} - iw^{(2)}/u^{(1)} - iu^{(2)}>_H = [<w^{(1)}/u^{(1)}>_H + <w^{(2)}/u^{(2)}>_H]$$
$$+i[<w^{(1)}/u^{(2)}>_H - <w^{(2)}/u^{(1)}>_H]$$

and

$$<\overline{w/u}>_H=<\overline{w^{(1)}+iw^{(2)}/u^{(1)}+iu^{(2)}}>_H$$

$$=\left[<w^{(1)}/u^{(1)}>_H+<w^{(2)}/u^{(2)}>_H\right]-i\left[<w^{(1)}/u^{(2)}>_H-<w^{(2)}/u^{(1)}>_H\right].$$

From our assumption, it follows that:

$$<\overline{w}/\overline{u}>_H=\overline{<w/u>_H}.$$

Hence

$$<e_\nu/\overline{U}-u>_H=\overline{<\overline{e_\nu}/U-\overline{u}>_H}=\overline{<\overline{e_\nu}/u>_H-<e_\nu/\overline{u}>_H}=0,$$

whenever $\nu=0,1,2,...$ By completeness of the system $\{e_\nu:\nu=0,1,2,...\}$, we therefore obtain $\overline{U}=u$, which ends the *Proof* of *Part (a)*, since

$$T_u(g)=<g/U>_H=<g/\overline{u}>_H \quad \text{for all } g\in H.$$

In particular, for $g(\cdot)=G_m(\cdot,z)$, we have

$$T_u(G_m(x,z))=\langle G_m(\cdot,z)/\overline{u(\cdot)}\rangle_H=\langle\sum_{j=0}^{m}\sigma_j^{(m)}(z)\overline{e_j(\cdot)}/\overline{u(\cdot)}\rangle_H$$

$$=\langle\left(\sum_{j=0}^{m}\overline{e_j(\cdot)}\sum_{k=0}^{m}\frac{K_X(z,\pi_{m,k})}{e_j(\pi_{m,k})}\right)/\overline{u(\cdot)}\rangle_H$$

$$=\overline{\langle\overline{u(\cdot)}/\left(\sum_{k=0}^{m}K_X(z,\pi_{m,k})\sum_{k=0}^{m}\frac{\overline{e_j(\cdot)}}{e_j(\pi_{m,k})}\right)\rangle_H}$$

$$=\langle u(\cdot)/\overline{\left(\sum_{k=0}^{m}K_X(z,\pi_{m,k})\sum_{j=0}^{m}\frac{\overline{e_j(\cdot)}}{e_j(\pi_{m,k})}\right)}\rangle_H$$

$$=\langle u(\cdot)/\overline{D_m(\cdot,z)}\rangle_H\ ,$$

where we have used the notation

$$D_m(x,z)=\sum_{k=0}^{m} K_X(z,\pi_{m,k})\sum_{j=0}^{m}\frac{\overline{e_j(x)}}{\overline{e_j(\pi_{m,k})}}.$$

The *Proof* is now complete.

Since $T_u(G_m(x,\cdot))\in H$ whenever $u\in H$, an application of the *Closed Graph Theorem* shows that the operator

$$T(G_m(x,\cdot)):H\to H:\ u(z)\mapsto T_u(G_m(x,z))=<u(\cdot)/\overline{D_m(\cdot,z)}>_H$$

is continuous. This operator is called *the generalized Padé-type operator for* H. The continuity property of the generalized Padé-type operator gives us interesting convergence results:

Theorem 3.4.8. *Under the assumptions of Theorem 3.4.7 and if the sequence* $\{u\in H\ :\nu=0,1,2,...\}$ *converges to* u *with respect to the* $\|\cdot\|_H-$*norm, we have*

$$\lim_{\nu\to\infty}T_{u_\nu}(G_m(x,\cdot))=T_u(G_m(x,\cdot)),$$

in the $\|\cdot\|_H-$*norm.*

Corollary 3.4.9. *Under the assumptions of Theorem 3.4.7 and if the series of functions*

$$u=\sum_{\nu=0}^{\infty}a_\nu\,u_\nu\ \ (a_\nu\in\mathrm{C},\ u_\nu\in H)$$

converges with respect to the $\|\cdot\|_H-$*norm,*

$$T_u(G_m(x,\cdot)) = \sum_{\nu=0}^{\infty} a_\nu T_{u_\nu}(G_m(x,.))$$

in the $\|\cdot\|_H$ *-norm.*

One may also define generalized Padé-type approximation to any linear operator (bounded or not)

$$F \ : H \mapsto H .$$

To do so, observe that the operator

$$T_{F(*)}(K_X(\cdot, x)): \ H \to H \ \ : u \mapsto T_{F(u)}(K_X(\cdot, x))$$

coincides with F, in the sense that

$$T_{F(u)}(K_X(z, x)) = [F(u)](z), \quad \text{for all } u \in H \text{ and all } z \in X .$$

Definition 3.4.10. *The linear operator*

$$T_{F(*)}(G_m(x,\cdot)) : H \to H : u \mapsto T_{F(u)}(G_m(x,\cdot)) = \sum_{j=0}^{m} \langle F(u) / e_j \rangle_H \ \sigma_j^{(m)}(\cdot)$$

is called a generalized a Padé-type approximant to the operator F, *with generating system* M_{m+1} .

In analogy to the preceding cases, the computation of a generalized Padé-type approximant $T_{F(*)}(G_m(x,\cdot))$ to F requires the knowledge of the functions

$$\sigma_0^{(m)}(\cdot), \sigma_1^{(m)}(\cdot), \ldots, \sigma_m^{(m)}(\cdot)$$

resulting from the equations

$$(H_j) \qquad \sigma_j^{(m)}(\cdot)=\sum_{k=0}^{m}\frac{K_X(\cdot,\pi_{m,k})}{\overline{e_j(\pi_{m,k})}} \qquad (j=0,1,2,\ldots,m),$$

and of the linear bounded functionals $< F(*)/e_0 >_H,\ldots,< F(*)/e_m >_H$:

$$(F_j) \qquad < F(*)/e_j >_H :\ H \to \mathbb{C} : g \mapsto < F(g)/e_j >_H \qquad (j=0,1,2,\ldots,m)$$

Remark 3.4.11. There are many cases where the functionals $< F(*)/e_j >_H$ can be handled by means of the spectral theorem. If, for example, F is in a closed normal subalgebra A of the Banach algebra of all bounded linear operators on H containing the identity operator, then there exists a unique *resolution of the identity* μ on the Borel subsets of the maximal ideal space Δ for A satisfying

$$< F(g)/e_j >_H = \int_{\Delta} \hat{\mathrm{F}}\ d\mu_{(g.e_j)} \qquad (g\in H,\ j=0,1,2,\ldots)$$

where $\hat{\mathrm{F}}$ is the *Gelfand Transform* of F; if F is self-adjoint, then there exists a unique resolution of the identity ρ on the Borel subsets of the real line, such that

$$< F(g)/e_j >_H = \int_{-\infty}^{\infty} t\, d\rho_{(g,e_j)}(t) \qquad (g\in H,\ j=0,1,2,\ldots).$$

From *Theorem 3.4.3.(a)*, it follows that the error of a generalized Padé-type approximation to F is the operator

$$T_{F(*)}(G_m(x,\cdot))-F(*)=\sum_{\nu=0}^{\infty}\left[\sum_{\substack{j=0\\(j\neq\nu)}}^{m}\langle F(*)/e_j\rangle_H\sum_{k=0}^{m}\frac{\overline{e_\nu(\pi_{m,k})}}{\overline{e_j(\pi_{m,k})}}\right]e_\nu(\cdot).$$

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A natural question which now arises is the following. Suppose we are given a sequence of generalized Padé-type approximants to the operator F. How can we tell whether this sequence converges to F ?

A reasonable and satisfactory answer to our question results directly from *Theorem 3.4.6*:

Theorem 3.4.12. *Let S be a subset of H. Let also*

$$M = \left(\pi_{m,k}\right)_{m\geq 0, 0\leq k\leq m}$$

be an infinite triangular matrix, with elements $\pi_{m,k} \in \mathrm{X}$ such that for any $m \geq 0$

$$\pi_{m,k} \neq \pi_{m,k'} \ \left(if \ \ k \neq k'\right),$$

$$\pi_{m,k} \notin \left(\bigcup_{0\leq j\leq m} Ker\overline{e_j(\cdot)}\right) \quad \left(if \ \ k \leq m\right)$$

and

$$\det\left[\overline{e_j\left(\pi_{m,k}\right)}\right]_{k,j} \neq 0 .$$

Suppose F is bounded. If

$$[\lim_{m\to\infty} \sum_{\substack{j=0 \\ (j\neq\nu)}}^{m} e_j(\cdot) \sum_{k=0}^{m} \frac{\overline{e_\nu\left(\pi_{m,k}\right)}}{\overline{e_j\left(\pi_{m,k}\right)}}] \in F(S)^{\perp} \qquad for\ every\ \nu = 0,1,2,...,$$

then there exists a subsequence $\{m_i : i = 0,1,2,...\}$ of $\{m \geq 0\}$ such that

$$\lim_{i\to\infty} T_{F(*)}\left(G_{m_i}(x,\cdot)\right) = F(*) \quad in\ S,$$

where all the limits are considered with respect to the $\|\cdot\|_H -norm$.

3.5. Applications

3.5.1 On Painlevé' s Theorem in Several Variables

The mapping properties of the Bergman kernel function have a central role in the study of bianalytic maps. It is well known that the statement of a *Riemann Mapping Theorem* in several complex variables must be quite different than in one variable. Chern and Moser built on the pioneering work of Poincaré and E. Cartan to produce a complete set of differential-geometric boundary invariants which must be preserved under a bianalytic map between smooth strictly pseudoconvex domains in $\mathbb{C}^n$ ([32]). In order to see that the Chern-Moser invariants are preserved under a bianalytic mapping, it is important to know that a bianalytic map between smooth strictly pseudoconvex domains must extend smoothly to the boundary. A fundamental result dealing with the C^∞ extension to the boundary of bianalytic maps between smooth strictly pseudoconvex domains was proved in 1974 by Fefferman ([59]). This result is classical in one complex variable, but in several variables it had been a major outstanding conjecture for many years. The first *Proof* in the one variable setting seems to be due to Painlevé ([115]). Other proofs were given by Kellogg and Warschawski (see [87] and [124]). One would like to adapt the proof of this result to more general situations, however many obstacles present themselves. The main purpose of this section is to propose an extension of *Painlevé's Theorem* in the case of arbitrary open subsets of $\mathbb{C}^n$, by using generalized Padé-type approximants.

Let $\Omega \neq \varnothing$ be a bounded open subset of $\mathbb{C}^n$. The subspace $OL^2(\Omega):=O(\Omega)\cap L^2(\Omega)$ is closed in the Hilbert space $L^2(\Omega)$ and hence is itself a Hilbert space. The evaluation map $OL^2(\Omega)\to\mathbb{C}: f\mapsto f(w)$ is a continuous $\mathbb{C}$-linear functional, whenever $w\in\Omega$. By the *Riesz Representation Theorem*, there exists a unique element $K_\Omega(\cdot,w)\in OL^2(\Omega)$ such that

$$f(w)=\int_\Omega f(\zeta)\overline{K_\Omega(\zeta,w)}dV(\zeta):=\langle f,K_\Omega(\cdot,w)\rangle$$

for all $f \in OL^2(\Omega)$. Here, $\langle \cdot,\cdot \rangle$ denotes the inner product in $L^2(\Omega)$ and $\|\cdot\|_2$ the corresponding norm. Remind that $K_\Omega(z,w)$ is called *the Bergman kernel function in* Ω as a function of z. The Bergman kernel function is analytic in z, conjugate analytic in w and satisfies $K_\Omega(z,w) = \overline{K_\Omega(w,z)}$ (cf. *Paragraph 3.2.1*). There is a bounded orthogonal projection, of norm 1, $P_\Omega : L^2(\Omega) \to OL^2(\Omega)$ called *the Bergman projection of* Ω, satisfying

$$P_\Omega(f)(\cdot) = \int_\Omega K_\Omega(\cdot,w) f(w) dV(w) \qquad f \in L^2(\Omega).$$

The property that the Bergman projection operator preserves differentiability up to the boundary can be used in the study of boundary regularity of bianalytic maps. *The open set* Ω *is said to satisfy condition* (R_d), for some $d \in \mathbb{N}$, if there is an integer s with $d+s>0$ such that the Bergman projection operator is a bounded map from $C^{d+s}(\overline{\Omega})$ into $C^d(\overline{\Omega})$, that is

$$\sum_{\substack{a\in \mathbb{N}_0^n \\ |a|\le d}} \sup_{z\in\overline{\Omega}} \left| D_z^{(a)} \int_\Omega K_\Omega(z,w) f(w) dV(w) \right| \le c_d \sum_{\substack{a\in \mathbb{N}_0^n \\ |a|\le d+s}} \sup_{z\in\overline{\Omega}} \left| D_z^{(a)} f(z) \right|,$$

($f \in C^{d+s}(\overline{\Omega})$) for some constant $c_d < \infty$. *The open set* Ω *is said to satisfy condition* (R), if it satisfies (R_d) for any $d \in \mathbb{N}$. One of *Bell's Theorems* says that *a bianalytic map between bounded pseudoconvex domains in* $\mathbb{C}^n$ *with* C^∞ *smooth boundary extends smoothly to the boundary as soon as at least one of the domains satisfies condition* (R) ([9]). Another important program in this direction has been initiated by Ligocka in [97]. The single most general contribution to come from this program is the discovery of the following:

Theorem 3.5.1. *If* Ω_1 *and* Ω_2 *are bounded pseudoconvex domains in* $\mathbb{C}^n$ *with boundaries of class* C^d *which satisfy condition* (R_d), *then every bianalytic mapping of* Ω_1 to Ω_2 *is in* $C^d(\overline{\Omega_1})$.

Condition (R) holds for many domains of $\mathbb{C}^n$ (for example, every smoothly bounded complete Reinhardt domain satisfies condition (R) ([124])). But on the other hand, Barrett found a smoothly bounded not-pseudoconvex domain in $\mathbb{C}^2$ which does not have property (R_d) for any $d \in \mathrm{N}$([7]). It should be mentioned that, since the Bergman projection P_Ω and the $\overline{\vartheta}$ -Neumann operator N are related via *Kohn's Formula* : $P_\Omega = I - \overline{\vartheta}^* N \overline{\vartheta}$ (,where $\overline{\vartheta}^*$ is the formal adjoint of the operator $\overline{\vartheta}$), whenever the $\overline{\vartheta}$ – Neumann operator associated to a domain satisfies global regularity estimates that domain satisfies condition (R). Kohn has shown in his paper [85] that the $\overline{\vartheta}$ – Neumann operator N satisfies these estimates in a variety of domains. Among these domains are smoothly bounded strictly pseudoconvex domains ([69]) and bounded pseudoconvex domains with real-analytic boundary ([86], [124]). Below, we shall give sufficient conditions for the extension of *Painlevé's classical Theorem* in $\mathbb{C}^n$. These considerations seem to be theoretical, but they succeed in eliminating both differential-geometry and subelliptic estimates for the $\overline{\vartheta}$ – Neumann problem and, on the other hand, they connect bianalytic extension problems with approximation and interpolation theory.

Let Ω be any bounded, non empty, open subset of $\mathbb{C}^n$ and let $\{\varphi_j : j = 0,1,2,...\}$ be an orthonormal basis for $OL^2(\Omega)$. Choose an infinite triangular matrix

$$\mathrm{M} = (\pi_{m,k})_{m \geq 0, 0 \leq k \leq m}$$

such that for any $m \geq 0$ there holds

$$\pi_{m,k} \in \Omega \ (k \leq m),$$

$$\pi_{m,k} \neq \pi_{m,k'} \ (k \neq k'),$$

$$\pi_{m,k} \notin \bigcup_{0 \leq j \leq m} Ker\overline{\varphi_j} \ \ (k \leq m),$$

and

$$\det\left[\overline{\varphi_j(\pi_{m,k})}\right]_{k,j} \neq 0 .$$

Consider the associated generalized Padé-type approximation sequence to Bergman's projection P_Ω:

$$\left\{T_{P_\Omega(\cdot)}(G_m(x,\cdot)):\ \ m = 0,1,2,\ldots\right\}.$$

In what follows, we shall assume that, whenever $w \in \Omega$ is fixed, the series

$$(C_d) \qquad \sum_{\nu=0}^{\infty} \varphi_\nu(\cdot)\,\overline{\varphi_\nu(w)}\ \ converges\ \ to \text{ to } K_\Omega(\cdot, w) \text{ in } C^d\left(\overline{\Omega}\right);$$

then, it is immediately seen that $K_\Omega(\cdot, w) \in C^d\left(\overline{\Omega}\right)$ and therefore

$$T_{P_\Omega\left(C^d\left(\overline{\Omega}\right)\right)}(G_m(x,\cdot)) \subset C^d\left(\overline{\Omega}\right).$$

In other words, the subspace $C^d\left(\overline{\Omega}\right)$ is an invariant subspace of the generalized Padé-type approximation operators.

Under the general presupposition (C_d), the restriction operators $T_{P_\Omega(*)}(G_m(x,\cdot))/_{C^d\left(\overline{\Omega}\right)}$ are continuous with respect to the topology induced by the norm $\|\cdot\|_{C_d\left(\overline{\Omega}\right)}$ of $C^d\left(\overline{\Omega}\right)$. In fact, for every $f \in C^d\left(\overline{\Omega}\right)$, there holds

$$\left\|T_{P_\Omega(f)}(G_m(x,\cdot))\right\|_{C^d\left(\overline{\Omega}\right)} \leq \sum_{j=0}^{m}\left|\langle P_\Omega(f), \varphi_j\rangle\right| \ \sum_{k=0}^{m}\frac{1}{\left|\varphi_j(\pi_{m,k})\right|}\ \left\|K_\Omega(\cdot, \pi_{m,k})\right\|_{C^d\left(\overline{\Omega}\right)}$$

$$\leq \sum_{j=0}^{m}\left\|P_\Omega(f)\right\|_2 \left\|\varphi_j\right\|_2 \sum_{k=0}^{m}\frac{1}{\left|\varphi_j(\pi_{m,k})\right|}\left\|K_\Omega(\cdot, \pi_{m,k})\right\|_{C^d\left(\overline{\Omega}\right)} \quad \text{(by \textit{Schwarz's Inequality})}$$

$$\leq \sum_{j=0}^{m}\|f\|_2 \ \sum_{k=0}^{m}\frac{1}{\left|\varphi_j(\pi_{m,k})\right|}\ \left\|K_\Omega(\cdot, \pi_{m,k})\right\|_{C^d\left(\overline{\Omega}\right)} \quad \left(\text{since}\left\|P_\Omega\right\| = 1 \text{ and} \left\|\varphi_j\right\|_2 = 1\right)$$

$$\leq \left\{\sum_{j=0}^{m}\sum_{k=0}^{m}\frac{1}{\left|\varphi_j(\pi_{m,k})\right|}\left\|K_\Omega(\cdot, \pi_{m,k})\right\|_{C^d\left(\overline{\Omega}\right)}\right\}\cdot\|f\|_2$$

$$\leq c_{\Omega}\left\{\sum_{j=0}^{m}\sum_{k=0}^{m}\frac{1}{\left|\varphi_j\left(\pi_{m,k}\right)\right|}\left\|K_{\Omega}\left(\cdot,\pi_{m,\kappa}\right)\right\|_{C^d\left(\overline{\Omega}\right)}\right\}\left\|f\right\|_{C^d\left(\overline{\Omega}\right)},$$

for some positive constant $c_{\Omega}<\infty$ depending only on Ω. Further, we have the

Theorem 3.5.2. *Let* $d\in\mathrm{N}_0=\mathrm{N}\bigcup\{0\}$. *Assume that* Ω *verifies Property* $\left(C_d\right)$ *and that there is a complete orthonormal basis* $\left\{\varphi_\nu\in C^d\left(\overline{\Omega}\right):\nu=0,1,2,...\right\}$ *for* $OL^2(\Omega)$ *and an infinite triangular matrix* $\mathrm{M}=\left(\pi_{m,k}\right)_{m\geq 0,0\leq k\leq m}$ *consisting of points* $\pi_{m,k}$ *in* Ω *such that*

$$\pi_{m,k}\neq\pi_{m,k'}\ \ (k\neq k'),\ \pi_{m,k}\notin\bigcup_{0\leq j\leq m}Ker\overline{\varphi_j}\ \ (k\leq m),\det\left[\overline{\varphi_j\left(\pi_{m,k}\right)}\right]_{k,j}\neq 0\,,$$

and

$$\lim_{m\to\infty}\sum_{j=0}^{m}\varphi_j(x)\left[\sum_{k=0}^{m}\frac{\varphi_\nu\left(\pi_{m,k}\right)}{\varphi_j\left(\pi_{m,k}\right)}\right]=\varphi_\nu(x)$$

for every $x\in\Omega$ *and every* $\nu=0,1,2,...$ *Then,* Ω *satisfies condition* $\left(C_d\right)$.

Proof. Let $f\in C^d\left(\overline{\Omega}\right)$. As it is pointed out in *Paragraph 3.2.3*,

$$P_{\Omega}(f)(z)=T_{P_{\Omega}(f)}\left(K_{\Omega}(z,x)\right).$$

Since $K_{\Omega}(z,\cdot)=\overline{K_{\Omega}(\cdot,z)}\in C^d\left(\overline{\Omega}\right)$, we see that

$$P_{\Omega}(f)\in C^d\left(\overline{\Omega}\right)$$

and therefore

$$\left\|P_{\Omega}(f)(\cdot)-T_{P_{\Omega}(f)}\left(G_m(x,\cdot)\right)\right\|_{C^d\left(\overline{\Omega}\right)}=\left\|T_{P_{\Omega}(f)}\left(K_{\Omega}(\cdot,x)-G_m(x,\cdot)\right)\right\|_{C^d\left(\overline{\Omega}\right)}.$$

By the integral representation of $T_{P_{\Omega}(f)}$, it holds

$$\left\| P_\Omega(f)(\cdot) - T_{P_\Omega(f)}\left(G_m(x,\cdot)\right)\right\|_{C^d(\overline{\Omega})} = \left\| \int_\Omega P_\Omega(f)(x)\left[K_\Omega(\cdot,x) - G_m(x,\cdot)\right] dV(x)\right\|_{C^d(\overline{\Omega})}$$

$$= \sum_{\substack{a\in \mathbb{N}_0^{2n} \\ |a|\le d}} \sup_{z\in\overline{\Omega}} \left| D_z^{(a)} \int_\Omega P_\Omega(f)(x)\left[K_\Omega(z,x) - G_m(x,z)\right] dV(x)\right|$$

$$\le \sum_{\substack{a\in \mathbb{N}_0^{2n} \\ |a|\le d}} \left\|P_\Omega(f)\right\|_2 \sup_{z\in\overline{\Omega}} \left\| D_z^{(a)}\left[K_\Omega(z,\cdot) - G_m(\cdot,z)\right]\right\|_2$$

$$= \left\|P_\Omega(f)\right\|_2 \sum_{\substack{a\in \mathbb{N}_0^{2n} \\ |a|\le d}} \sup_{z\in\overline{\Omega}} \left\| D_z^{(a)}\left[K_\Omega(z,\cdot) - G_m(\cdot,z)\right]\right\|_2 .$$

Now, observe that

$$D_z^{(a)} K_\Omega(z,x) = D_z^{(a)} \sum_{\nu=0}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(x)} = \sum_{\nu=0}^{\infty} \overline{\varphi_\nu(x)}\, D_z^{(a)} \varphi_\nu(z)$$

and

$$D_z^{(a)} G_m(z,x) = D_z^{(a)} \sum_{j=0}^{m}\sum_{k=0}^{m} \frac{K_\Omega(z,\pi_{m,k})}{\varphi_j(\pi_{m,k})}\overline{\varphi_j(x)} = D_z^{(a)} \sum_{j=0}^{m}\sum_{k=0}^{m} \frac{\overline{\varphi_j(x)}}{\varphi_j(\pi_{m,k})} \sum_{\nu=0}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(\pi_{m,k})}$$

$$= \sum_{\nu=0}^{\infty} \left\{ \sum_{j=0}^{m}\sum_{k=0}^{m} \frac{\overline{\varphi_j(x)}\;\overline{\varphi_\nu(\pi_{m,k})}}{\varphi_j(\pi_{m,k})} \right\} D_z^{(a)} \varphi_\nu(z).$$

It follows that

$$\left\| P_\Omega(f)(\cdot) - T_{P_\Omega(f)}\left(G_m(x,\cdot)\right)\right\|_{C^d(\overline{\Omega})}$$

$$\le \left\|P_\Omega(f)\right\|_2 \sum_{\substack{a\in \mathbb{N}_0^{2n} \\ |a|\le d}} \sup_{z\in\overline{\Omega}} \left[\int_\Omega \left| \sum_{\nu=0}^{\infty} \left\{ \overline{\varphi_\nu(x) - \sum_{j=0}^{m}\sum_{k=0}^{m} \frac{\varphi_j(x)\,\varphi_\nu(\pi_{m,k})}{\varphi_j(\pi_{m,k})}} \right\} D_z^{(a)} \varphi_\nu(z) \right|^2 dV(x) \right]^{\frac{1}{2}} .$$

Setting

$$H_m^{(a)}(x,z) := \sum_{\nu=0}^{\infty} \left\{ \overline{\varphi_\nu(x) - \sum_{j=0}^{m} \varphi_j(x) \sum_{k=0}^{m} \frac{\varphi_\nu(\pi_{m,k})}{\varphi_j(\pi_{m,k})}} \right\} D_z^{(a)} \varphi_\nu(z),$$

it is enough to show, that, for any $a \in \mathbb{N}_0^{2n}$ with $|a| \le d$ and any $\varepsilon > 0$, there exists a $M = M(a,\varepsilon)$ such that

$$\left| H_m^{(a)}(x,z) \right| < \varepsilon,$$

for any $m \ge 0$, every $z \in \overline{\Omega}$ and almost all $x \in \Omega$. Since

$$\lim_{m\to\infty} \sum_{j=0}^{m} \varphi_j(x) \sum_{k=0}^{m} \frac{\varphi_\nu(\pi_{m,k})}{\varphi_j(\pi_{m,k})} = \varphi_\nu(x) \qquad \text{for almost all } x \in \Omega,$$

it suffices to obtain the uniform convergence on $\overline{\Omega}$ of the series $H_m^{(a)}(x,\cdot)$, whenever $x \in \Omega$, $m \ge 0$ and $a \in \mathbb{N}_0^{2n}$ with $|a| \le d$. But, this is true because of *Property* (C_d). In fact, for each $x \in \Omega$ fixed, each $m \ge 0$ and each $a \in \mathbb{N}_0^{2n}$ fixed ($|a| \le d$), we have

$$\sum_{\nu=0}^{\infty} \left\{ \overline{\varphi_\nu(x) - \sum_{j=0}^{m} \varphi_j(x) \sum_{k=0}^{m} \frac{\varphi_\nu(\pi_{m,k})}{\varphi_j(\pi_{m,k})}} \right\} D_z^{(a)} \varphi_\nu(\cdot)$$

$$= \sum_{\nu=0}^{\infty} \overline{\varphi_\nu(x)}\, D_z^{(a)} \varphi_\nu(\cdot) - \sum_{\nu=0}^{\infty} \sum_{j=0}^{m} \overline{\varphi_j(x)} \sum_{k=0}^{m} \frac{\overline{\varphi_\nu(\pi_{m,k})}}{\overline{\varphi_j(\pi_{m,k})}} D_z^{(a)} \varphi_\nu(\cdot)$$

$$= D_z^{(a)} \left(\sum_{\nu=0}^{\infty} \varphi_\nu(\cdot)\, \overline{\varphi_\nu(x)} \right) - \sum_{j=0}^{m} \overline{\varphi_j(x)} \sum_{k=0}^{m} \frac{1}{\overline{\varphi_j(\pi_{m,k})}} \sum_{\nu=0}^{\infty} \overline{\varphi_\nu(\pi_{m,k})}\, D_z^{(a)} \varphi_\nu(\cdot)$$

$$= D_z^{(a)} K_\Omega(\cdot,x) - \sum_{j=0}^{m} \overline{\varphi_j(x)} \sum_{k=0}^{m} \frac{1}{\overline{\varphi_j(\pi_{m,k})}} D_z^{(a)} \left(\sum_{\nu=0}^{\infty} \varphi_\nu(\cdot)\, \overline{\varphi_\nu(\pi_{m,k})} \right)$$

$$= D_z^{(a)} K_\Omega(\cdot,x) - \sum_{j=0}^{m} \overline{\varphi_j(x)} \sum_{k=0}^{m} \frac{1}{\overline{\varphi_j(\pi_{m,k})}} D_z^{(a)} K_\Omega(\cdot,\pi_{m,k}),$$

with uniform convergence on $\overline{\Omega}$ of any series participating in the above equalities. This proves the *Theorem*.

3.5.2. Numerical Examples

Example 3.5.3. It is well known that the function

$$f_a(t) = e^{at} \qquad (-\pi < t < \pi \quad \text{and} \quad a \neq 0),$$

has the following Fourier series representation

$$\mathrm{F_a(t)} = F_a(t) = \frac{e^{a\pi} - e^{-a\pi}}{\pi}\left[\frac{1}{2a} + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}}{\nu^2 + a^2}(a\cos(\nu t) - \nu\sin(\nu t))\right]$$

on the interval $-\pi < t < \pi$. It holds

$$f_a(t) = F_a(t) \qquad \text{for any } t \in (-\pi, \pi).$$

(However,

$$F_a(\pm\pi) = \cosh(a\pi) \neq e^{\pm a\pi} = f_a(\pm\pi)$$

and therefore we must consider the following $2\pi-$periodic extension of f_a on $[-\pi, \pi]$:

$$\tilde{f}_a(t) = \begin{cases} e^{at}, & \textit{if} \quad -\pi < t < \pi \\ \cosh(a\pi), & \textit{if} \quad t = \pm\pi \end{cases}$$

and then approximate $\tilde{f}_a(t)$ in the Padé-type sense.) Evidently, for every $t \in (-\pi, \pi)$, we have

$$f_a(t) = e^{at} = F_a(t) = \frac{e^{a\pi} - e^{-a\pi}}{\pi}\left[\frac{1}{2a} + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}}{\nu^2 + a^2}(a\cos(\nu t) - \nu\sin(\nu t))\right]$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{2a\pi}\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}\left(e^{a\pi} - e^{-a\pi}\right)}{2\pi\left(\nu^2 + a^2\right)}(a + i\nu)e^{i\nu t} + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu}\left(e^{a\pi} - e^{-a\pi}\right)}{2\pi\left(\nu^2 + a^2\right)}(a - i\nu)e^{-i\nu t}$$

$$=\sum_{\nu=-\infty}^{\infty}\frac{(-1)^{\nu}\left(e^{a\pi}-e^{-a\pi}\right)(a+i\nu)}{2\pi\left(\nu^{2}+a^{2}\right)}e^{i\nu t} \qquad \left(\text{or} \right.$$

$$\left. =\sum_{\nu=-\infty}^{\infty}\frac{(-1)^{\nu}\left(e^{a\pi}-e^{-a\pi}\right)}{2\pi(a-i\nu)}e^{i\nu t} \right).$$

Define the $\mathbb{C}$-linear functional $T_{f_a}:\mathbb{P}(\mathbb{C})\to\mathbb{C}$ associated with f by

$$T_{f_a}(x^{\nu})=c_{\nu}^{(a)}:=\frac{(-1)^{\nu}(e^{a\pi}-e^{-a\pi})(a+i\nu)}{2\pi(\nu^{2}+a^{2})} \qquad (\nu=0,1,2,...).$$

Given any matrix

$$\mathrm{M}=\left(\pi_{m,k}\right)_{m\geq 0,0\leq k\leq m}$$

with complex entries $\pi_{m,k}\in D\,(\Leftrightarrow\left|\pi_{m,k}\right|<1)$, then, for any $m\geq 0$, a Padé-type approximant to $f_a(t)$ is a function

$$\operatorname{Re} m/m+1)_{f_a}(t)=2\operatorname{Re}T_{f_a}\left(Q_m\left(x,e^{it}\right)\right)-c_0^{(a)}$$

$$=2.\operatorname{Re}\left[\frac{e^{-it}.T_{f_a}\left(\dfrac{V_{m+1}(e^{-it})-V_{m+1}(x)}{e^{-it}-x}\right)}{V_{m+1}(e^{-it})}\right]-c_0^{(a)}$$

where $Q_m\left(x,e^{it}\right)$ is the unique interpolation polynomial of $\left(1-xe^{it}\right)^{-1}$ at $\left(\pi_{m,0},e^{it}\right),\left(\pi_{m,1},e^{it}\right),...,\left(\pi_{m,m},e^{it}\right)$ and where

$$V_{m+1}(x):=\gamma\prod_{k=0}^{m}\left(x-\pi_{m,k}\right)$$

is the generating polynomial of this approximation ($\gamma\in\mathbb{C}-\{0\}$). For information about Padé-

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type approximation to a $2\pi-$periodic L^2-function, one may consult [43].

We will consider several different cases.

(a). Choose $m=4$ and $\pi_{4,0}=\pi_{4,1}=\pi_{4,2}=\pi_{4,3}=\pi_{4,4}=0$; then

- $V_5(x)=x^5$,
- $e^{-it}T_{f_a}\left(\dfrac{V_5(e^{-it})-V_5(x)}{e^{-it}-x}\right)=e^{-it}T_{f_a}\left(x^4+x^3e^{-it}+x^2e^{-2it}+xe^{-3it}+e^{-4it}\right)$

$$=e^{-it}.\left(C_4^{(a)}+e^{-it}C_3^{(a)}+e^{-2it}C_2^{(a)}+e^{-3it}C_1^{(a)}+e^{-4it}C_0^{(a)}\right)$$

$$=\frac{e^{-it}(e^{a\pi}-e^{-a\pi})}{2\pi}\left[\frac{a+4i}{16+a^2}-e^{-it}\frac{a+3i}{9+a^2}+e^{-2it}\frac{a+2i}{4+a^2}-e^{-3it}\frac{a+i}{1+a^2}-e^{-4it}\frac{1}{a}\right],$$

- $V_5(e^{-it})=e^{-5it}$,

and

$$\mathrm{Re}(4/5)_{f_a}=2\,\mathrm{Re}\left[\frac{e^{a\pi}-e^{-a\pi}}{2\pi.e^{-4it}}\left(\frac{a+4i}{16+a^2}-e^{-it}\frac{a+3i}{9+a^2}+e^{-2it}\frac{a+2i}{4+a^2}-e^{-3it}\frac{a+i}{1+a^2}+e^{-it}\frac{1}{a}\right)\right]$$

$$-\frac{e^{a\pi}-e^{-a\pi}}{2\pi a}$$

$$=\frac{e^{a\pi}-e^{-a\pi}}{\pi}\left[\frac{\mathrm{Re}\{(a+4i)e^{4it}\}}{16+a^2}-\frac{\mathrm{Re}\{(a+3i)e^{3it}\}}{9+a^2}+\frac{\mathrm{Re}\{(a+2i)e^{2it}\}}{4+a^2}-\frac{\mathrm{Re}\{(a+i)e^{it}\}}{1+a^2}+\frac{1}{2a}\right].$$

Since

$$\mathrm{Re}\{(a+4i)e^{4it}\}=a\cos(4t)-4\sin(4t),\quad \mathrm{Re}\{(a+3i)e^{3it}\}=a\cos(3t)-3\sin(3t),$$

and

$$\mathrm{Re}\{(a+2i)e^{2it}\}=a\cos(2t)-2\sin(2t),\quad \mathrm{Re}\{(a+i)e^{it}\}=a\cos t-\sin t,$$

it follows that

$$\mathrm{Re}(4/5)_{f_a} = \frac{e^{a\pi} - e^{-a\pi}}{\pi}\left[\frac{a\cos(4t) - 4\sin(4t)}{16 + a^2} - \frac{a\cos(3t) - 3\sin(3t)}{9 + a^2}\right.$$

$$\left. + \frac{a\cos(2t) - 2\sin(2t)}{4 + a^2} - \frac{a.\cos t - \sin t}{1 + a^2} + \frac{1}{2a}\right],$$

that is

$$\mathrm{Re}(4/5)_{f_a}(t) = \frac{e^{a\pi} - e^{-a\pi}}{\pi}\left[\frac{1}{2a} + \sum_{\nu=1}^{4}\frac{(-1)^{\nu}}{\nu^2 + a^2}(a\cos(\nu t) - \nu\sin(\nu t))\right] \qquad ([43]).$$

In other words, if $m = 4$ and $\pi_{4,0} = \pi_{4,1} = \pi_{4,2} = \pi_{4,3} = \pi_{4,4} = 0$, then the Padé-type approximant $\mathrm{Re}(4/5)_{f_a}(t)$ is nothing else than the trigonometric polynomial formed by summing exactly the first five terms in the Fourier series $F_a(t)$ of $f_a(t)$. Unfortunately, this choice is not very successful because of the failure of the corresponding approximation in some trivial (but characteristic) cases. If, for example, $t = 0$, then

$$\mathrm{Re}(4/5)_{f_a}(0) = \frac{a\left(e^{a\pi} - e^{-a\pi}\right)}{\pi}\left[\frac{430 - 110a^2 - 10a^4}{(16 + a^2)(9 + a^2)(4 + a^2)(1 + a^2)} + \frac{1}{2a^2}\right],$$

and for $a = 1$ we obtain

$$\mathrm{Re}(4/5)_{f_1}(0) \approx 5.0116 \text{ (while in such a case } f_1(0) = 1\text{)}.$$

Similarly, for $a = -1$,

$$\mathrm{Re}(4/5)_{f_{-1}}(0) \approx 5.0116 \text{ (,while } f_{-1}(0)\text{. equals 1)}.$$

Further, if $t = 1$, then

$$\mathrm{Re}(4/5)_{f_a}(t) = \frac{e^{a\pi} - e^{-a\pi}}{\pi}\left[\frac{a(-0.5) + 3.4641016}{16 + a^2} - \frac{a(-1)}{9 + a^2} + \frac{a(-0.5) - 1.7320508}{4 + a^2}\right.$$

$$\left. - \frac{a(0.5) - 0.8660254}{1 + a^2} + \frac{1}{2a}\right],$$

and,for $a=1$, we have

$$\mathrm{Re}(4/5)_{f_1}(1)\approx 7.3521621\,[0.1743589+0.1-0.4464101+0.1830127+0.5] = 3.756617$$

(while in such a case $f_1(1)=e\approx 2.7182818$).

(b). Choose $m=3$ and $\pi_{3,0}=\pi_{3,1}=\pi_{3,2}=0$, $\pi_{3,3}=-\frac{i}{2}$. Then

- $V_4(x)=x^4+i\dfrac{x^3}{2}$,
- $e^{-it}T_{f_a}\left(\dfrac{V_4(e^{-it})-V_4(x)}{e^{-it}-x}\right)=e^{-it}T_{f_a}\left(x^3+\left[e^{-it}+\frac{i}{2}\right]x^2+\left[e^{-2it}+i\,\frac{e^{-it}}{2}\right]x\right.$

$$\left.+\left[e^{-3it}+i\frac{e^{-2it}}{2}\right]\right)$$

$$=e^{-it}\left(C_3^{(a)}+\left[e^{-it}+\frac{i}{2}\right]C_2^{(a)}+\left[e^{-2it}+i\,\frac{e^{-it}}{2}\right]C_1^{(a)}+\left[e^{-3it}+i\,\frac{e^{-2it}}{2}\right]C_0^{(a)}\right)$$

$$=\frac{e^{a\pi}-e^{-a\pi}}{2\pi}e^{-4it}\left(\left[\frac{1}{a}\right]+\left[\frac{i}{2a}-\frac{a+i}{1+a^2}\right]e^{it}+\left[\frac{a+2i}{4+a^2}+\frac{1-ai}{2+2a^2}\right]e^{it}\right.$$

$$\left.+\left[\frac{-a-3i}{9+a^2}\right]+\left[\frac{-2+ai}{8+2a^2}\right]e^{3it}\right),$$

- $V_4\left(e^{-it}\right)=e^{-4it}\left(1+i\dfrac{e^{it}}{2}\right)$.

Therefore,

$$\mathrm{Re}(3/4)_{f_a}(t)=\frac{e^{a\pi}-e^{-a\pi}}{\pi}\left\{\mathrm{Re}\left(\frac{\left[\frac{1}{a}\right]+\left[\frac{i}{2a}-\frac{a+i}{1+a^2}\right]e^{it}+\left[\frac{a+2i}{4+a^2}+\frac{1-ai}{2+2a^2}\right]e^{2it}}{1+i\,\frac{e^{it}}{2}}\right.\right.$$

$$+\frac{\left[\frac{-a-3i}{9+a^2}+\frac{-2+ai}{8+2a^2}\right]e^{3it}}{1+i\frac{e^{it}}{2}}\Bigg)-\frac{1}{2a}\Bigg\}.$$

Let us give a more explicit form for $\mathrm{Re}(3/4)_{f_a}(t)$. Since

$$1+i\frac{e^{it}}{2}=\left(1-\frac{\sin t}{2}\right)+i\frac{\cos t}{2},$$

$$\left(1-\frac{\sin t}{2}\right)^2+\left(\frac{\cos t}{2}\right)^2=\frac{5}{4}\sin t,$$

$$\left[\frac{i}{2a}-\frac{a+i}{1+a^2}\right]e^{it}=\left[-\frac{a}{1+a^2}\cos t-\left(\frac{1}{2a}-\frac{1}{1+a^2}\right)\sin t\right]$$

$$+i\left[-\frac{a}{1+a^2}\sin t-\left(\frac{1}{2a}-\frac{1}{1+a^2}\right)\cos t\right],$$

$$\left[\frac{a+2i}{4+a^2}+\frac{1-ai}{2+2a^2}\right]e^{2it}=\left[\left(\frac{a}{4+a^2}+\frac{1}{2+2a^2}\right)\cos 2t-\left(\frac{2}{4+a^2}-\frac{a}{2+2a^2}\right)\sin 2t\right]$$

$$+i\left[\left(\frac{a}{4+a^2}+\frac{1}{2+2a^2}\right)\sin 2t+\left(\frac{2}{4+a^2}-\frac{a}{2+2a^2}\right)\cos 2t\right],$$

$$\left[\frac{-a-3i}{9+a^2}+\frac{-2+ai}{8+2a^2}\right]e^{3it}=\left[\left(\frac{-a}{9+a^2}+\frac{-2}{8+2a^2}\right)\cos 3t-\left(\frac{a}{8+2a^2}-\frac{3}{9+a^2}\right)\sin 3t\right]$$

$$+i\left[\left(\frac{-a}{9+a^2}+\frac{-2}{8+2a^2}\right)\sin 3t+\left(\frac{a}{8+2a^2}-\frac{3}{9+a^2}\right)\cos 3t\right],$$

we have

$$\mathrm{Re}(3/4)_{f_a}(t)$$

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$$
\begin{aligned}
&= \frac{1}{\frac{5}{4} - \sin t} \frac{e^{a\pi} - e^{-a\pi}}{\pi} \left\{ \left[\frac{1}{a} - \frac{a}{1+a^2} \cos t - \left(\frac{1}{2a} - \frac{1}{1+a^2} \right) \sin t + \left(\frac{a}{4+a^2} + \frac{1}{2+2a^2} \right) \cos 2t \right.\right. \\
&- \left(\frac{2}{4+a^2} + \frac{a}{2+2a^2} \right) \sin 2t + \left(\frac{-a}{9+a^2} + \frac{-2}{8+2a^2} \right) \cos 3t - \left(\frac{a}{8+2a^2} - \frac{3}{9+a^2} \right) \sin 3t \Bigg] \left[1 - \frac{\sin t}{2} \right] \\
&+ \left[\frac{-a}{1+a^2} . \sin t + \left(\frac{1}{2a} - \frac{1}{1+a^2} \right) . \cos t + \left(\frac{a}{4+a^2} + \frac{1}{2+2a^2} \right) \sin 2t + \left(\frac{2}{4+a^2} - \frac{a}{2+2a^2} \right) \cos 2t \right. \\
&\left.\left. + \left(\frac{-a}{9+a^2} + \frac{-2}{8+2a^2} \right) \sin 3t + \left(\frac{a}{8+2a^2} - \frac{3}{9+a^2} \right) \cos 3t \right] \left[\frac{\cos t}{2} \right] - \frac{\frac{5}{4} - \sin t}{2a^2} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{5 - 4\sin t} \frac{e^{a\pi} - e^{-a\pi}}{\pi} \left\{ \left[\frac{5}{8a} - \frac{1}{2+2a^2} \right] + \left[-\frac{1}{2a} + \frac{1}{1+a^2} + \frac{a}{8+2a^2} + \frac{1}{4+4a^2} \right] \sin t \right. \\
&+ \left[-\frac{5a}{4+4a^2} + \frac{1}{4+a^2} \right] \cos t + \left[-\frac{2}{4+a^2} + \frac{1}{2+2a^2} \right] \sin 2t \\
&+ \left[\frac{9a}{32+8a^2} + \frac{1}{2+2a^2} - \frac{3}{36+4a^2} \right] \cos 2t + \left[-\frac{a}{8+2a^2} + \frac{3}{9+a^2} \right] \sin 3t \\
&\left. + \left[-\frac{a}{9+a^2} - \frac{1}{4+a^2} \right] \cos 3t + \left[-\frac{a}{18+2a^2} - \frac{1}{8+2a^2} \right] \sin 4t \right\}.
\end{aligned}
$$

In particular, for $a = 1$ there holds

$$
\mathrm{Re}(3/4)_{f_1}(t) = \frac{1}{5 - 4\sin t} \frac{e^{\pi} - e^{-\pi}}{10\pi}
$$

$$
\{15 + 9\sin t - 17\cos t - 8\sin 2t + 14\cos 2t + 8\sin 3t - 12\cos 3t - 6\sin 4t\},
$$

and if $t = 0$, then

$$
\mathrm{Re}(3/4)_{f_1}(0) = 0 \text{ (while } f_1(0) = 1\text{).}
$$

If $t = 1$, then

$$\mathrm{Re}(3/4)_{f_1}(1) \approx 8.4068037 \text{ (while } f_1(1) = e \approx 2.7182818\text{)}.$$

As in the preceding case, these disappoint approximate results attest the failure of the choice

$$\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = 0 \text{ and } \pi_{3,3} = -\frac{i}{2}.$$

(c). Let $m = 3$. We choose the zeros of the Tchebycheff polynomials

$$\mathrm{TCH}_m(X) = \cos(m\ Arc\cos X)$$

divided by $\sqrt{\pi}$ as interpolation nodes, i.e. $\pi_{3,k} = \frac{1}{\sqrt{\pi}}\cos\left[\frac{2k+1}{7}\pi\right]$:

$$\pi_{3,0} = \frac{1}{\sqrt{\pi}}\cos\frac{\pi}{7}, \quad \pi_{3,1} = \frac{1}{\sqrt{\pi}}\cos\frac{3\pi}{7}, \quad \pi_{3,2} = \frac{1}{\sqrt{\pi}}\cos\frac{5\pi}{7}, \quad \pi_{3,3} = \frac{1}{\sqrt{\pi}}\cos\pi.$$

Then,

- $V_4(x) \approx \left(x - \frac{1}{\sqrt{\pi}}0.9009688\right)\left(x - \frac{1}{\sqrt{\pi}}0.2225209\right)\left(x + \frac{1}{\sqrt{\pi}}0.6234898\right)\left(x + \frac{1}{\sqrt{\pi}}\right)$

$= x^4 + 0.282x^3 - 0.318x^2 - 0.067x + 0.012$,

- $V_4(e^{-it}) \approx e^{-4it}\left(1 + 0.282e^{it} - 0.318e^{2it} - 0.067e^{3it} + 0.012e^{4it}\right)$

$$= \frac{e^{-it}}{V_4(e^{-it})} T_{f_a}\left(\frac{V_4(e^{-it}) - V_4(x)}{e^{-it} - x}\right) = \frac{e^{-it}}{V_4(e^{-it})} T_f\Big(x^3 + [e^{-it} + 0.282]x^2$$

$$+ [e^{-2it} + 0.282\,e^{-it} - 0.318]\,x + [e^{-3it} + 0.282\,e^{-2it} - 0.318\,e^{-it} - 0.037]\Big)$$

$$= \frac{c_0 + [c_1 + 0.282\,c_0]\,e^{it} + [c_2 + 0.282\,c_1 - 0.318\,c_0]\,e^{2it}}{1 + 0.282\,e^{it} - 0.318\,e^{2it} - 0.067\,e^{3it} + 0.012\,e^{4it}}$$

$$+ \frac{[c_3 + 0.282\,c_2 - 0.318\,c_1 - 0.067\,c_0]\,e^{3it}}{1 + 0.282\,e^{it} - 0.318\,e^{2it} - 0.067\,e^{3it} + 0.012\,e^{4it}}$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{2\pi}$$

$$\left\{ \frac{\frac{1}{a}+\left[-\frac{a+i}{1+a^2}+0.282\frac{1}{a}\right]e^{it}+\left[\frac{a+2i}{4+a^2}-0.282\frac{a-i}{1+a^2}-0.318\frac{1}{a}\right]e^{2it}}{1+0.282\,e^{it}-0.318\,e^{2it}-0.067\,e^{3it}+0.012\,e^{4it}} \right.$$

$$\left. +\frac{\left[-\frac{a+3i}{9+a^2}+0.282\frac{a-2i}{4+a^2}+0.318\frac{a+i}{1+a^2}-0.067\frac{1}{a}\right]e^{3it}}{1+0.282\,e^{it}-0.318\,e^{2it}-0.067\,e^{3it}+0.012\,e^{4it}} \right\}.$$

It follows that

$$\mathrm{Re}(3/4)_{f_a}(t)=2\,\mathrm{Re}\left[\frac{e^{-it}}{V_4(e^{-it})}T_{f_a}\left(\frac{V_4(e^{-it})-V_4(x)}{e^{-it}-x}\right)\right]-c_0$$

$$=2\,\mathrm{Re}\left[\frac{e^{-it}}{V_4(e^{-it})}T_{f_a}\left(\frac{V_4(e^{-it})-V_4(x)}{e^{-it}-x}\right)\right]-\frac{e^{a\pi}-e^{-a\pi}}{2\pi a}$$

$$\approx\frac{e^{a\pi}-e^{-a\pi}}{\pi}\{(1+0.282\cos t-0.318\cos 2t-0.067\cos 3t+0.012\cos 4t)^2$$

$$+(1+0.282\sin t-0.318\sin 2t-0.067\sin 3t+0.012\sin 4t)^2\}^{-1}$$

$$\left\{\left[\frac{1.185137}{a}-\frac{0.21363\,a}{1+a^2}-\frac{0.336894\,a}{4+a^2}+\frac{0.067\,a}{9+a^2}\right]\right.$$

$$+\left[\frac{1.521358}{1+a^2}-\frac{0.51188}{4+a^2}-\frac{0.99}{9+a^2}\right]\sin t$$

$$+\left[\frac{0.426456}{a}-\frac{0.839938}{1+a^2}+\frac{0.128708.a}{4+a^2}+\frac{0.306\,a}{9+a^2}\right]\cos t$$

$$+\left[\frac{0.25594}{1+a^2}-\frac{2.135048}{4+a^2}+\frac{0.846}{9+a^2}\right]\sin 2t$$

$$+\left[-\frac{0.677604}{a}-\frac{0.12708\ a}{1+a^2}+\frac{1.091524\ a}{4+a^2}-\frac{0.282\ a}{9+a^2}\right]\cos 2t$$

$$+\left[-\frac{0.33}{1+a^2}-\frac{0.564}{4+a^2}-\frac{3}{9+a^2}\right]\sin 3t$$

$$+\left[-\frac{0.130616}{a}+\frac{0.306\ a}{1+a^2}+\frac{0.282\ a}{4+a^2}-\frac{a}{9+a^2}\right]\cos 3t$$

$$+\left[\frac{0.012}{a}\right]\cos 4t\Bigg\}-\frac{e^{a\pi}-e^{-a\pi}}{2\pi a} \qquad ([43]).$$

In particular, for $a=1$, there holds

$$\mathrm{Re}(3/4)_{f_1}(t)\approx 7.352161\ \{(1+0.0282\ \cos t-0.318\ \cos 2t-0.067\ \cos 3t+0.012\ \cos 4t)^2+$$

$$(1+0.0282\ \sin t-0.318\ \sin 2t-0.067\ \sin 3t+0.012\ \sin 4t)^2\}^{-1}$$

$$\{[1.0176432]+[0.559303]\sin t+[0.0628286]\cos t+[-0.2144396]\sin 2t$$

$$+[-0.5518532]\cos 2t+[-0.5778]\sin 3t+[-0.021216]\cos 3t+[0.012]\cos 4t\}$$

$$-3.676081.$$

Thus,

if $t=0$,

then

$$\mathrm{Re}(3/4)_{f_1}(0)\approx 0.9455091 \quad (\text{, while } f_1(0)=e^0=1);$$

if $t=1$,

then

$$\mathrm{Re}(3/4)_{f_1}(1)\approx 2.8227598 \quad (\text{, while } f_1(1)=e\approx 2.7182818);$$

if $t=e$,

then

$$\mathrm{Re}(3/4)_{f_1}(e) \approx 15.968062 \quad (\text{, while } f_1(e) = e^e \approx 15.154261).$$

(However, if $t = \sqrt{3}$, then

$$\mathrm{Re}(3/4)_{f_1}(\sqrt{3}) \approx 7.6652958, \quad \text{while } f_1(\sqrt{3}) = e^{\sqrt{3}} = 5.6522335,$$

and if $t = \dfrac{\pi}{2}$, then

$$\mathrm{Re}(3/4)_{f_1}\left(\frac{\pi}{2}\right) \approx 5.7613728, \quad \text{while } f_1\left(\frac{\pi}{2}\right) = 4.810477\,.)$$

(d). The above choice of the interpolation nodes (i.e.,

$$\pi_{3,k} = \frac{1}{\sqrt{\pi}}\cos\left[\frac{2k+1}{7}\pi\right] \quad ,k = 0,1,2,3)$$

seems to be satisfactory at least for the case $a = 1$. However, if $a = -1$, then

$$\mathrm{Re}(3/4)_{f_{-11}}(0) \approx 8.4191897 \quad (\text{while } f_{-1}(0) = e^{-10} \approx 1)$$

and if $a = 2$ then

$$\mathrm{Re}(3/4)_{f_2}(0) \approx 60.407038 \quad (\text{while } f_2(0) = 1).$$

On the other hand, for relative choices of the interpolation nodes, the corresponding approximations are not very successful.

If, for example, $m = 3$ and $\pi_{3,k}$ are simply the zeros of the Tchebycheff polynomials $\mathrm{TCH}_3(X) = \cos(3\,Arc\cos X)$ on $[-\pi, \pi]$, i.e.

$$\pi_{3,k} = \cos\left[\frac{2k+1}{7}\pi\right] \quad (k = 0,1,2,3),$$

then

$$\mathrm{Re}(3/4)_{f_1}(0) \approx -1.2760657 \ (\text{while } f_1(0) = 1).$$

If $m = 2$, and

$$\pi_{2,0} = 0,\ \pi_{2,1} = \sqrt{\frac{3}{5}}\pi \text{ and } \pi_{2,2} = \sqrt{\frac{3}{5}}\pi,$$

then

$$\mathrm{Re}(2/3)_{f_1}(0) = 4.1242233 \text{ (while } f_1(0) = 1\text{)}.$$

If $m = 4$ and $\pi_{4,k}$ $(k = 0,1,2,3,4)$ are the zeros of the Legendre polynomial

$$LEG_5(x) = x^5 + \frac{38}{9}\pi^2 x^3 - \frac{311}{105}\pi^4 x$$

on $[-\pi, \pi]$, then

$$\mathrm{Re}(4/5)_{f_a}(t)$$

$$= \frac{e^{a\pi} - e^{-a\pi}}{\pi} \frac{1}{(1 + 41.6716\cos 2t - 288.5165\cos 4t)^2 + (41.6716\sin 2t - 288.5165\sin 4t)^2}$$

$$\left\{\left[84979.256\frac{1}{a} - 11981.288\frac{a}{4+a^2} - 288.516\frac{a}{16+a^2}\right]\right.$$

$$+\left[13718.815\frac{1}{1+a^2} + 990.564\frac{1}{9+a^2}\right]\sin t + \left[10243.76\frac{1}{1+a^2} + 246.845\frac{1}{9+a^2}\right]\cos t$$

$$+\left[-4052088\frac{1}{4+a^2} - 166.686\frac{1}{16+a^2}\right]\sin 2t + \left[-23962.576\frac{1}{a} + 1449.011\frac{a}{4+a^2}\right.$$

$$\left. + 41.671\frac{a}{16+a^2}\right]\cos 2t$$

$$+\left[330.18\frac{1}{1+a^2} + 3\frac{1}{9+a^2}\right]\sin 3t + \left[246.85\frac{a}{1+a^2} - \frac{a}{9+a^2}\right]\cos 3t$$

$$\left. +\left[-4\frac{1}{16+a^2} - 83.343\frac{1}{4+a^2}\right]\sin 4t + \left[-577.033\frac{1}{a} + 41.6716\frac{1}{4+a^2} + \frac{a}{16+a^2}\right]\cos 4t\right\}$$

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$$-\frac{e^{a\pi}-e^{-a\pi}}{2\pi a}$$

(see [43]). So for $a=1$ and $t=0$ we have

$$\mathrm{Re}(4/5)_{f_1}(0)=4.0601491 \quad (\text{while } f_1(0)=1\ ).$$

For $a=-1$ and $t=0$ we have

$$\mathrm{Re}(4/5)_{f_{-1}}(0)=4.0601421 \quad (\text{while } f_{-1}(0)=1).$$

But, for $a=2$ and $t=0$ there holds

$$\mathrm{Re}(4/5)_{f_2}(0)=47.084848 \qquad (\text{while } f_2(0)=1).$$

(e). After these comments, it is obvious that the existence of the real parameter $a\neq 0$ may cause various perturbations in the behavior of our approximants and, for this reason, the investigation of a more efficacious and general conception is legitimate.

Without loss of generality, we will assume that

$$|a|<A<\infty.$$

Since $|t|<\pi$, our starting point will be the consideration of the analytic function

$$f:\Delta^1(0,A)\times\Delta^1(0,\pi)\to\mathbb{C}:(z_1,z_2)\mapsto f(z_1,z_2):=e^{z_1 z_2}.$$

Obviously,

$$f(a,t)=e^{at}=f_a(t) \text{ for every } a\in(-A,A) \text{ and for every } t\in(-\pi,\pi)$$

and hence, the approximation *in the generalized Padé-type sense* of the complex-valued function f gives several particular results for every real-valued function f_a: Since $f\in OL^2(\Delta^1(0,A)\times\Delta^1(0,\pi))$, we can also write

$$f(z_1,z_2)=\int_{\Delta^1(0,A)\times\Delta^1(0,\pi)} f(\zeta_1,\zeta_2)\,\overline{K_{\Delta^1(0,A)\times\Delta^1(0,\pi)}((\zeta_1,\zeta_2),(z_1,z_2))}\,dV(\zeta_1,\zeta_2)$$

for any $(z_1,z_2)\in\Delta^1(0,A)\times\Delta^1(0,\pi)$, where $K_{\Delta^1(0,A)\times\Delta^1(0,\pi)}$ is the Bergman kernel function on $\Delta^1(0,A)\times\Delta^1(0,\pi)$, i.e.

$$K_{\Delta^1(0,A)\times\Delta^1(0,\pi)}((\zeta_1,\zeta_2),(z_1,z_2))=\frac{1}{\pi^2}\frac{A\,\pi}{(A^2-\zeta_1\,\overline{z_1})^2.(\pi^2-\zeta_2\,\overline{z_2})^2}$$

(see *Theorem 3.2.2*), and where $dV(\zeta_1,\zeta_2)$ is the volume form of $\mathbb{C}^2$, that is

$$dV(\zeta_1,\zeta_2)=\left(\frac{1}{2}\right)^2 d\zeta_1\wedge d\overline{\zeta_1}\wedge d\zeta_2\wedge d\overline{\zeta_2}\,.$$

As it is well known an orthonormal basis for $OL^2(\Delta^1(0,A)\times\Delta^1(0,\pi))$ is the set of monomials

$$\left\{\left[\int_{\Delta^1(0,A)\times\Delta^1(0,\pi)}\left|\zeta_1^{j_1}\zeta_2^{j_2}\right|dV(\zeta_1,\zeta_2)\right]\zeta_1^{j_1}\zeta_2^{j_2}\,:\,j_1=0,1,2,\ldots,\,j_2=0,1,2,\ldots\right\}.$$

If $\zeta_1=x_1+i\,y_1$ and $\zeta_2=x_2+i\,y_2$, then

$$\begin{aligned}dV(\zeta_1,\zeta_2)&=\left(\frac{1}{2}\right)^2 d\zeta_1\wedge d\overline{\zeta_1}\wedge d\zeta_2\wedge d\overline{\zeta_2}\\&=\frac{1}{4}\left([dx_1+idy_1]\wedge[dx_1-idy_1]\wedge[dx_2+idy_2]\wedge[dx_2-idy_2]\right)\\&=\frac{1}{4}\left([-idx_1\wedge dy_1+idy_1\wedge dx_1]\wedge[-idx_2\wedge dy_2+idy_2\wedge dx_2]\right)\\&=\frac{1}{4}\left([-2i\,dx_1\wedge dy_1]\wedge[-2i\,dx_2\wedge dy_2]\right)\\&=dx_1\wedge dy_1\wedge dx_2\wedge dy_2\ ,\end{aligned}$$

and therefore

$$\int_{\Delta^1(0,A)\times\Delta^1(0,\pi)}|\,\zeta_1^{j_1}\zeta_2^{j_2}\,|^2\,dV(\zeta_1,\zeta_2)=\int_{x_1^2+y_1^2<A^2}\left[x_1^2+y_1^2\right]^{j_1}dx_1\wedge dy_1\int_{x_2^2+y_2^2<\pi^2}\left[x_2^2+y_2^2\right]^{j_2}dx_2\wedge dy_2$$

$$= \int_{x_1^2+y_1^2<A^2} d\left\{\left(-y_1\int_0^1 t\left[x_1^2t^2+y_1^2t^2\right]^{j_1}dt\right)dx_1+\left(x_1\int_0^1 t\left[x_1^2t^2+y_1^2t^2\right]^{j_1}dt\right)dy_1\right\}$$

$$\int_{x_2^2+y_2^2<\pi^2} d\left\{\left(-y_2\int_0^1 t\left[x_2^2t^2+y_2^2t^2\right]^{j_2}dt\right)dx_2+\left(x_2\int_0^1 t\left[x_2^2t^2+y_2^2t^2\right]^{j_2}dt\right)dy_2\right\}$$

(by the *Proof* of *Poincaré's Lemma*)

$$= \int_{x_1^2+y_1^2<A^2} d\left\{\left(-y_1\frac{\left[x_1^2+y_1^2\right]^{j_1}}{2j_1+2}\right)dx_1+\left(x_1\frac{\left[x_1^2+y_1^2\right]^{j_1}}{2j_1+2}\right)dy_1\right\}$$

$$\int_{x_2^2+y_2^2<\pi^2} d\left\{\left(-y_2\frac{\left[x_2^2+y_2^2\right]^{j_1}}{2j_2+2}\right)dx_2+\left(x_2\frac{\left[x_2^2+y_2^2\right]^{j_2}}{2j_2+2}\right)dy_2\right\}$$

$$=\left\{\int_{x_1^2+y_1^2=A^2}\frac{-y_1\left[x_1^2+y_1^2\right]^{j_1}dx_1+x_1\left[x_1^2+y_1^2\right]^{j_1}dy_1}{2j_1+2}\right\}$$

$$\left\{\int_{x_2^2+y_2^2=\pi^2}\frac{-y_2\left[x_2^2+y_2^2\right]^{j_2}dx_2+x_2\left[x_2^2+y_2^2\right]^{j_2}dy_2}{2j_2+2}\right\}$$

(by *Stokes' Formula*)

$$=\frac{A^{2j_1+2}\,\pi^{2j_2+2}}{(2j_1+2)(2j_2+2)}\left\{\int_{x_1^2+y_1^2=A^2}\frac{-y_1dx_1+x_1dy_1}{x_1^2+y_1^2}\right\}\left\{\int_{x_2^2+y_2^2=\pi^2}\frac{-y_2dx_2+x_2dy_2}{x_2^2+y_2^2}\right\}$$

$$=\frac{A^{2j_1+3}\,\pi^{2j_2+5}}{(j_1+1)(j_2+1)}.$$

It follows that an orthonormal basis for $OL^2(\Delta^1(0,A)\times\Delta^1(0,\pi))$ is the set of monomials

$$\left\{\varphi_{j_1,j_2}(z_1,z_2)=\frac{\sqrt{j_1+1}\sqrt{j_2+1}}{\sqrt{A\pi}\,A^{j_1+1}\pi^{j_2+2}}\,z_1^{j_1}z_2^{j_2}\ :\ j_1=0,1,2,\ldots,\ j_2=0,1,2,\ldots\right\}.$$

The Fourier series expansion of $f(z_1,z_2)=e^{z_1 z_2}$ with respect to this basis is given by

$$f(z_1,z_2)=e^{z_1 z_2} = \sum_{j_1,j_2=0}^{\infty} a_{j_1,j_2}^{(f)} \, \varphi_{j_1,j_2}(z_1 z_2),$$

with

$$a_{j_1,j_2}^{(f)} = \int_{\Delta^1(0,A)\times\Delta^1(0,\pi)} f(\zeta_1,\zeta_2)\,\overline{\varphi_{j_1,j_2}(\zeta_1,\zeta_2)}\,dV(\zeta_1,\zeta_2).$$

From the uniqueness of the Taylor series expansion for $f(z_1,z_2)=e^{z_1 z_2}$, it follows that

$$a_{j_1,j_2}^{(f)} = \frac{\sqrt{A\pi}\;A^{j_1+1}\pi^{j_2+2}}{\sqrt{j_1+1}\,\sqrt{j_2+1}\;j_1!\,j_2!} \qquad (j_1\ge 0,\, j_2\ge 0).$$

Thus, an orthonormal basis for $OL^2(\Delta^1(0,A)\times\Delta^1(0,\pi))$ is the set

$$\left\{\varphi_{j_1,j_2}(z_1,z_2)=\frac{\sqrt{j_1+1}\sqrt{j_2+1}}{\sqrt{A\pi}\;A^{j_1+1}\pi^{j_2+2}}\,z_1^{j_1}z_2^{j_2} : j_1=0,1,2,\ldots,\, j_2=0,1,2,\ldots\right\},$$

and the sequence of Fourier coefficients of f with respect to this basis is the set

$$\left\{a_{j_1,j_2}^{(f)} = \frac{\sqrt{A\pi}\;A^{j_1+1}\pi^{j_2+2}}{\sqrt{j_1+1}\sqrt{j_2+1}\;j_1!\,j_2!};\quad j_1=0,1,2,\ldots,\, j_2=0,1,2,\ldots\right\}.$$

Consider now the infinite array:

$$\begin{array}{cccc}
\varphi_{0,0} & \varphi_{0,1} & \varphi_{0,2} & \varphi_{0,3}\cdots \\
\varphi_{1,0} & \varphi_{1,1} & \varphi_{1,2} & \varphi_{1,3}\cdots \\
\varphi_{2,0} & \varphi_{2,1} & \varphi_{2,2} & \varphi_{2,3}\cdots \\
\varphi_{3,0} & \varphi_{3,1} & \varphi_{3,2} & \varphi_{3,3}\cdots \\
\multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

The array contains all the elements of the basis. As indicated by the arrows, these elements can be arranged in a sequence:

$$\varphi_0 := \varphi_{0,0}, \varphi_1 := \varphi_{1,0}, \varphi_2 := \varphi_{0,1}, \varphi_3 := \varphi_{2,0}, \varphi_4 := \varphi_{1,1}, \varphi_5 := \varphi_{0,2}, \varphi_6 := \varphi_{3,0}, \varphi_7 := \varphi_{2,1}, \ldots$$

Indicatively, we observe that the first ten elements of this sequence are

$$\varphi_0(z_1, z_2) = A^{-\frac{3}{2}}\pi^{-\frac{5}{2}},$$

$$\varphi_1(z_1, z_2) = \sqrt{2}A^{-\frac{5}{2}}\pi^{-\frac{5}{2}}z_1,$$

$$\varphi_2(z_1, z_2) = \sqrt{2}A^{-\frac{3}{2}}\pi^{-\frac{7}{2}}z_2,$$

$$\varphi_3(z_1, z_2) = \sqrt{3}A^{-\frac{7}{2}}\pi^{-\frac{5}{2}}z_1^2,$$

$$\varphi_4(z_1, z_2) = 2A^{-\frac{5}{2}}\pi^{-\frac{7}{2}}z_1 z_2,$$

$$\varphi_5(z_1, z_2) = \sqrt{3}A^{-\frac{3}{2}}\pi^{-\frac{9}{2}}z_2^2,$$

$$\varphi_6(z_1, z_2) = 2A^{-\frac{9}{2}}\pi^{-\frac{5}{2}}z_1^3,$$

$$\varphi_7(z_1, z_2) = \sqrt{6}A^{-\frac{7}{2}}\pi^{-\frac{7}{2}}z_1^2 z_2,$$

$$\varphi_8(z_1, z_2) = \sqrt{6}A^{-\frac{5}{2}}\pi^{-\frac{9}{2}}z_1 z_2^2,$$

$$\varphi_9(z_1, z_2) = 2A^{-\frac{3}{2}}\pi^{-\frac{11}{2}}z_2^3.$$

Similarly, the Fourier coefficients $a_{j_1,j_2}^{(f)}$ of f can be arranged in a sequence:

$$a_0^{(f)} := a_{0,0}^{(f)}, a_1^{(f)} := a_{1,0}^{(f)}, a_2^{(f)} := a_{0,1}^{(f)}, a_3^{(f)} := a_{2,0}^{(f)}, a_4^{(f)} := a_{1,1}^{(f)}, a_5^{(f)} := a_{0,2}^{(f)},$$

$$a_6^{(f)} := a_{3,0}^{(f)} \ldots$$

Indicatively, the first ten elements of this sequence are

$$a_0^{(f)} = A^{\frac{3}{2}}\pi^{\frac{5}{2}},$$

$$a_1^{(f)} = \frac{A^{\frac{5}{2}}\pi^{\frac{5}{2}}}{\sqrt{2}},$$

$$a_2^{(f)} = \frac{A^{\frac{3}{2}}\pi^{\frac{7}{2}}}{\sqrt{2}},$$

$$a_3^{(f)} = \frac{A^{\frac{7}{2}}\pi^{\frac{5}{2}}}{2\sqrt{3}},$$

$$a_4^{(f)} = \frac{A^{\frac{5}{2}}\pi^{\frac{7}{2}}}{2},$$

$$a_5^{(f)} = \frac{A^{\frac{3}{2}}\pi^{\frac{9}{2}}}{2\sqrt{3}},$$

$$a_6^{(f)} = \frac{A^{\frac{9}{2}}\pi^{\frac{5}{2}}}{12},$$

$$a_7^{(f)} = \frac{A^{\frac{7}{2}}\pi^{\frac{7}{2}}}{2\sqrt{6}},$$

$$a_8^{(f)} = \frac{A^{\frac{5}{2}}\pi^{\frac{9}{2}}}{2\sqrt{6}},$$

$$a_9^{(f)} = \frac{A^{\frac{3}{2}}\pi^{\frac{11}{2}}}{12}.$$

For any $m \geq 0$, choose a finite set of pair-wise distinct points

$$\left\{\pi_{m,0} = \left(\pi_{m,0}^{(1)}, \pi_{m,0}^{(2)}\right), \pi_{m,1} = \left(\pi_{m,1}^{(1)}, \pi_{m,1}^{(2)}\right), \ldots, \pi_{m,m} = \left(\pi_{m,m}^{(1)}, \pi_{m,m}^{(2)}\right)\right\} \subset \Delta^1(0, A) \times \Delta^1(0, \pi),$$

in such a way that the determinant

$$\det\left[\overline{\varphi_j\left(\pi_{m,k}\right)}\right]_{k,j} = \begin{vmatrix} \overline{\varphi_0\left(\pi_{m,0}\right)} & \overline{\varphi_1\left(\pi_{m,0}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,0}\right)} \\ \overline{\varphi_0\left(\pi_{m,1}\right)} & \overline{\varphi_1\left(\pi_{m,1}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,1}\right)} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \overline{\varphi_0\left(\pi_{m,m}\right)} & \overline{\varphi_1\left(\pi_{m,m}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,m}\right)} \end{vmatrix}$$

is different from zero.

If $j \le m$, we set

$$c_j^{(m)}(z_1,z_2) := \sum_{k=0}^{m} \frac{K_{\Delta^1(0,A)\times\Delta^1(0,\pi)}((z_1,z_2),(\pi_{m,k}^{(1)},\pi_{m,k}^{(2)}))}{\overline{\varphi_j(\pi_{m,k})}}$$

$$= \sum_{k=0}^{m} \frac{A}{\pi\left(A^2 - z_1\overline{\pi_{m,k}^{(1)}}\right)^2\left(\pi^2 - z_2\overline{\pi_{m,k}^{(2)}}\right)^2 \overline{\varphi_j\left(\pi_{m,k}\right)}}.$$

The sum

$$\left(GPTA/m\right)_f\left(z_1,z_2\right) := \sum_{j=0}^{m} a_j^{(f)}\, c_j^{(m)}(z_1,z_2)$$

$$= \frac{A}{\pi}\sum_{j=0}^{m} a_j^{(f)} \sum_{k=0}^{m} \frac{1}{\left(A^2 - z_1.\overline{\pi_{m,k}^{(1)}}\right)^2\left(\pi^2 - z_2.\overline{\pi_{m,k}^{(2)}}\right)^2 \overline{\varphi_j\left(\pi_{m,k}\right)}}$$

is then a generalized Padé-type approximant to the complex-valued function

$$f\left(z_1,z_2\right) = e^{z_1 z_2}$$

into the bounced open $\Delta^1\left(0,A\right)\times\Delta^1\left(0,\pi\right)\subset\mathbb{C}^2$.

If, for example, $m = 5$, then there holds

$$\left(GPTA/5\right)_f\left(z_1,z_2\right)$$

$$= A^4\pi^4\left\{\frac{1}{\left(A^2 - z_1\overline{\pi_{5,0}^{(1)}}\right)^2\left(\pi^2 - z_2\overline{\pi_{5,0}^{(2)}}\right)^2} + \ldots + \frac{1}{\left(A^2 - z_1\overline{\pi_{5,5}^{(1)}}\right)^2\left(\pi^2 - z_2\overline{\pi_{5,5}^{(2)}}\right)^2}\right\}$$

$$+\frac{A^6\pi^4}{2}\left\{\frac{1}{\left(A^2-z_1\overline{\pi_{5,0}^{(1)}}\right)^2\left(\pi^2-z_2.\overline{\pi_{5,0}^{(2)}}\right)^2\overline{\pi_{5,0}^{(1)}}}+\ldots+\frac{1}{\left(A^2-z_1\overline{\pi_{5,5}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{5,5}^{(2)}}\right)^2\overline{\pi_{5,5}^{(1)}}}\right\}$$

$$+\frac{A^4\pi^6}{2}\left\{\frac{1}{\left(A^2-z_1\overline{\pi_{5,0}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{5,0}^{(2)}}\right)^2\overline{\pi_{5,0}^{(2)}}}+\ldots+\frac{1}{\left(A^2-z_1\overline{\pi_{5,5}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{5,5}^{(2)}}\right)^2\overline{\pi_{5,5}^{(2)}}}\right\}$$

$$+\frac{A^8\pi^4}{6}\left\{\frac{1}{\left(A^2-z_1\overline{\pi_{5,0}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{5,0}^{(2)}}\right)^2\left(\overline{\pi_{5,0}^{(1)}}\right)^2}+\ldots+\frac{1}{\left(A^2-z_1\overline{\pi_{5,5}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{5,5}^{(2)}}\right)^2\left(\overline{\pi_{5,5}^{(1)}}\right)^2}\right\}$$

$$+\frac{A^6\pi^6}{4}\left\{\frac{1}{\left(A^2-z_1\overline{\pi_{5,0}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{5,0}^{(2)}}\right)^2\overline{\pi_{5,0}^{(1)}\pi_{5,0}^{(2)}}}+\ldots+\frac{1}{\left(A^2-z_1\overline{\pi_{5,5}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{5,5}^{(2)}}\right)^2\overline{\pi_{5,5}^{(1)}\pi_{5,5}^{(2)}}}\right\}$$

$$+\frac{A^4\pi^8}{6}\left\{\frac{1}{\left(A^2-z_1\overline{\pi_{5,0}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{5,0}^{(2)}}\right)^2\left(\overline{\pi_{5,0}^{(2)}}\right)^2}+\ldots+\frac{1}{\left(A^2-z_1\overline{\pi_{5,5}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{5,5}^{(2)}}\right)^2\left(\overline{\pi_{5,5}^{(2)}}\right)^2}\right\},$$

under the constraint:

$$\det\begin{pmatrix}
1 & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,0}^{(1)}}\right) & \frac{\sqrt{2}}{\pi}\left(\overline{\pi_{5,0}^{(2)}}\right) & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,0}^{(1)}}\right)^2 & \frac{2}{A\pi}\left(\overline{\pi_{5,0}^{(1)}}.\overline{\pi_{5,0}^{(2)}}\right) & \frac{\sqrt{3}}{\pi^2}\left(\overline{\pi_{5,0}^{(2)}}\right)^2 \\
1 & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,1}^{(1)}}\right) & \frac{\sqrt{2}}{\pi}\left(\overline{\pi_{5,1}^{(2)}}\right) & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,1}^{(1)}}\right)^2 & \frac{2}{A\pi}\left(\overline{\pi_{5,1}^{(1)}}.\overline{\pi_{5,1}^{(2)}}\right) & \frac{\sqrt{3}}{\pi^2}\left(\overline{\pi_{5,1}^{(2)}}\right)^2 \\
1 & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,2}^{(1)}}\right) & \frac{\sqrt{2}}{\pi}\left(\overline{\pi_{5,2}^{(2)}}\right) & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,2}^{(1)}}\right)^2 & \frac{2}{A\pi}\left(\overline{\pi_{5,2}^{(1)}}.\overline{\pi_{5,2}^{(2)}}\right) & \frac{\sqrt{3}}{\pi^2}\left(\overline{\pi_{5,2}^{(2)}}\right)^2 \\
1 & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,3}^{(1)}}\right) & \frac{\sqrt{2}}{\pi}\left(\overline{\pi_{5,3}^{(2)}}\right) & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,3}^{(1)}}\right)^2 & \frac{2}{A\pi}\left(\overline{\pi_{5,3}^{(1)}}.\overline{\pi_{5,3}^{(2)}}\right) & \frac{\sqrt{3}}{\pi^2}\left(\overline{\pi_{5,3}^{(2)}}\right)^2 \\
1 & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,4}^{(1)}}\right) & \frac{\sqrt{2}}{\pi}\left(\overline{\pi_{5,4}^{(2)}}\right) & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,4}^{(1)}}\right)^2 & \frac{2}{A\pi}\left(\overline{\pi_{5,4}^{(1)}}.\overline{\pi_{5,4}^{(2)}}\right) & \frac{\sqrt{3}}{\pi^2}\left(\overline{\pi_{5,4}^{(2)}}\right)^2 \\
1 & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,5}^{(1)}}\right) & \frac{\sqrt{2}}{\pi}\left(\overline{\pi_{5,5}^{(2)}}\right) & \frac{\sqrt{3}}{A}\left(\overline{\pi_{5,5}^{(1)}}\right)^2 & \frac{2}{A\pi}\left(\overline{\pi_{5,5}^{(1)}}.\overline{\pi_{5,5}^{(2)}}\right) & \frac{\sqrt{3}}{\pi^2}\left(\overline{\pi_{5,5}^{(2)}}\right)^2
\end{pmatrix}\neq 0.$$

As it pointed out in *Paragraph 3.2.3*, the best choice of the interpolation points

$$\pi_{5,0}, \pi_{5,1}, \pi_{5,2}, \pi_{5,3}, \pi_{5,4}, \pi_{5,5} \in \Delta^1(0, A) \times \Delta^1(0, \pi)$$

is an open and difficult optimization problem of several complex variables. However, in order to appreciate the effectiveness of the generalized Padé-type method, let us give a typical result about the approximation of the real function

$$f_a(t) = e^{at} \quad (a \neq 0).$$

Of course $f_a(t) = f(a,t)$ and in practice we can always take

$$A = |a|.$$

If $\varepsilon > 0, \zeta > 0$ and $\delta > 0$ are very small, then, by choosing

$$\pi_{5,k}^{(1)} \approx 0 \qquad \text{and} \qquad \pi_{5,k}^{(2)} \approx \pi \qquad (k = 0,1,2,3,4,5)$$

such that

$$\pi_{5,0}^{(1)} = \exp \frac{Log\left(-\dfrac{24}{A^2\pi^2} + i\,\varepsilon\right)\displaystyle\prod_{k=1}^{5} \left[\frac{Log\,\overline{\pi_{5,k}^{(2)}}}{Log\,\overline{\pi_{5,k}^{(1)}} + Log\,\overline{\pi_{5,k}^{(2)}}}\right] + Log\left(-\dfrac{\pi^2}{A^2} + i\,\delta\right)}{\displaystyle\prod_{k=1}^{5} Log\,\overline{\pi_{5,k}^{(1)}} + \prod_{k=1}^{5} Log\,\overline{\pi_{5,k}^{(2)}}},$$

$$\pi_{5,0}^{(2)} = \exp\left[\frac{Log\left(-\dfrac{24}{A^2\pi^2} + i\,\varepsilon\right)\displaystyle\prod_{k=1}^{5} \left[\frac{Log\,\overline{\pi_{5,k}^{(1)}}}{Log\,\overline{\pi_{5,k}^{(1)}} + Log\,\overline{\pi_{5,k}^{(2)}}}\right] - Log\left(-\dfrac{\pi^2}{A^2} + i\,\delta\right)}{\displaystyle\prod_{k=1}^{5} Log\,\overline{\pi_{5,k}^{(1)}} + \prod_{k=1}^{5} Log\,\overline{\pi_{5,k}^{(2)}}}\right],$$

and

$$\frac{\displaystyle\sum_{k=0}^{5} \left[\overline{\pi_{5,k}^{(1)}}\right]^{-2}}{\displaystyle\sum_{k=0}^{5} \left[\overline{\pi_{5,k}^{(2)}}\right]^{-2}} \approx \left(\frac{\pi^4}{A^4} + i\,\zeta\right),$$

we have

$$A^4\pi^4\sum_{k=0}^{5}\left(A^2-a\,\overline{\pi_{5,k}^{(1)}}\right)^{-2}\left(\pi^2-0\,\overline{\pi_{5,k}^{(2)}}\right)^{-2}$$

$$\approx -\frac{1}{4}A^6\pi^6\sum_{k=0}^{5}\left(A^2-a\,\overline{\pi_{5,k}^{(1)}}\right)^{-2}\left(\pi^2-0\,\overline{\pi_{5,k}^{(2)}}\right)^{-2}\left(\overline{\pi_{5,k}^{(1)}}\;\overline{\pi_{5,k}^{(2)}}\right)^{-1},$$

$$\frac{1}{6}A^8\pi^4\sum_{k=0}^{5}\left(A^2-a\,\overline{\pi_{5,k}^{(1)}}\right)^{-2}\left(\pi^2-0\,\overline{\pi_{5,k}^{(2)}}\right)^{-2}\left(\overline{\pi_{5,k}^{(1)}}\right)^{-2}$$

$$\approx -\frac{1}{6}A^4\pi^8\sum_{k=0}^{5}\left(A^2-a\,\overline{\pi_{5,k}^{(1)}}\right)^{-2}\left(\pi^2-0\,\overline{\pi_{5,k}^{(2)}}\right)^{-2}\left(\overline{\pi_{5,k}^{(2)}}\right)^{-2},$$

$$\frac{1}{2}A^6\pi^4\sum_{k=0}^{5}\left(A^2-a\,\overline{\pi_{5,k}^{(1)}}\right)^{-2}\left(\pi^2-0\,\overline{\pi_{5,k}^{(2)}}\right)^{-2}\left(\overline{\pi_{5,k}^{(1)}}\right)^{-1}$$

$$\approx -\frac{1}{2}A^4\pi^6\sum_{k=0}^{5}\left(A^2-a\,\overline{\pi_{5,k}^{(1)}}\right)^{-2}\left(\pi^2-0\,\overline{\pi_{5,k}^{(2)}}\right)^{-2}\left(\overline{\pi_{5,k}^{(2)}}\right)^{-1}$$

for any $a \neq 0$, which implies that

$$(GPTA/5)_f(a,0) \approx f_a(0), \qquad \text{for any } a>0.$$

(Here, the function $Log : \mathbb{C} -]\infty, 0) \to \mathbb{C}$ denotes the principal branch of the logarithm, that is

$$Logz := \log|z| + i\arg z, \text{ with } -\pi < \arg z < \pi.)$$

Furthermore, under the same constraint and if

$$\sum_{\substack{j=0\\(j\neq v)}}^{5} a_j^{(f)} \sum_{k=0}^{5} \frac{\overline{\varphi_v(\pi_{5,k})}}{\overline{\varphi_j(\pi_{5,k})}} = 0 \qquad (v=0,1,2,3,4,5),$$

or explicitly if

$$\sum_{k=0}^{5}\frac{1}{\left[\overline{\pi_{5,k}^{(1)}}\;\overline{\pi_{5,k}^{(2)}}\right]^2}\left\{2\pi^4\left[\overline{\pi_{5,k}^{(1)}}\right]^2+2A^4\left[\overline{\pi_{5,k}^{(2)}}\right]^2+3A^2\pi^2\left[\overline{\pi_{5,k}^{(1)}}\right]\left[\overline{\pi_{5,k}^{(2)}}\right]+6\pi^2\left[\overline{\pi_{5,k}^{(1)}}\right]^2\left[\overline{\pi_{5,k}^{(2)}}\right]^2\right.$$

$$+6A^2\left[\pi_{5,k}^{(1)}\right]\left[\pi_{5,k}^{(2)}\right]^2\Big\}=0,$$

$$\sum_{k=0}^{5}\frac{1}{\left[\pi_{5,k}^{(1)}\right]\left[\pi_{5,k}^{(2)}\right]^2}\Big\{2\pi^4\left[\pi_{5,k}^{(1)}\right]^2+2A^4\left[\pi_{5,k}^{(2)}\right]^2+3A^4\pi^4\left[\pi_{5,k}^{(1)}\right]\left[\pi_{5,k}^{(2)}\right]+6\pi^2\left[\pi_{5,k}^{(1)}\right]^2\left[\pi_{5,k}^{(2)}\right]$$

$$+12\left[\pi_{5,k}^{(1)}\right]^2\left[\pi_{5,k}^{(2)}\right]^2\Big\}=0,$$

$$\sum_{k=0}^{5}\frac{1}{\left[\pi_{5,k}^{(1)}\right]^2\left[\pi_{5,k}^{(2)}\right]}\Big\{2\pi^4\left[\pi_{5,k}^{(1)}\right]^2+2A^4\left[\pi_{5,k}^{(2)}\right]^2+3A^2\pi^2\left[\pi_{5,k}^{(1)}\right]\left[\pi_{5,k}^{(2)}\right]+6A^2\left[\pi_{5,k}^{(1)}\right]\left[\pi_{5,k}^{(2)}\right]^2$$

$$+12\left[\pi_{5,k}^{(1)}\right]^2\left[\pi_{5,k}^{(2)}\right]^2\Big\}=0,$$

$$\sum_{k=0}^{5}\frac{1}{\left[\pi_{5,k}^{(2)}\right]^2}\Big\{2\pi^4\left[\pi_{5,k}^{(1)}\right]^2+3A^2\pi^2\left[\pi_{5,k}^{(1)}\right]\left[\pi_{5,k}^{(2)}\right]+6\pi^2\left[\pi_{5,k}^{(1)}\right]^2\left[\pi_{5,k}^{(2)}\right]+6A^2\left[\pi_{5,k}^{(1)}\right]\left[\pi_{5,k}^{(2)}\right]^2$$

$$+12\left[\pi_{5,\kappa}^{(1)}\right]^2\left[\pi_{5,\kappa}^{(2)}\right]^2\Big\}=0,$$

$$\sum_{k=0}^{5}\frac{1}{\left[\pi_{5,k}^{(1)}\right]\left[\pi_{5,k}^{(2)}\right]}\Big\{6\pi^4\left[\pi_{5,k}^{(1)}\right]^2+2A^4\left[\pi_{5,k}^{(2)}\right]^2+6\pi^2\left[\pi_{5,k}^{(1)}\right]^2\left[\pi_{5,k}^{(2)}\right]+3A^2\left[\pi_{5,k}^{(1)}\right]\left[\pi_{5,k}^{(2)}\right]^2$$

$$+6\left[\pi_{5,\kappa}^{(1)}\right]^2\left[\pi_{5,\kappa}^{(2)}\right]^2\Big\}=0,$$

and

$$\sum_{k=0}^{5}\frac{1}{\left[\pi_{5,k}^{(1)}\right]^2}\Big\{2A^4\left[\pi_{5,k}^{(2)}\right]^2+3A^2\pi^2\left[\pi_{5,k}^{(1)}\right]\left[\pi_{5,k}^{(2)}\right]+6\pi^2\left[\pi_{5,k}^{(1)}\right]^2\left[\pi_{5,k}^{(2)}\right]+6A^2\left[\pi_{5,k}^{(1)}\right]\left[\pi_{5,k}^{(2)}\right]^2$$

$$+12\left[\pi_{5,k}^{(1)}\right]^2\left[\pi_{5,k}^{(2)}\right]^2\Big\}=0,$$

then the Fourier series expansion of $(GPTA/5)_f(z_1,z_2)$ with respect to the basis

$$\{\varphi_0(z_1,z_2),\varphi_1(z_1,z_2),\ldots\}$$

matches the Fourier series expansion of $f(z_1,z_2)$ (with respect to the same basis) up to the $(m+1)^{th}$ term. In such a case, the function $(GPTA/5)_f(z_1,z_2)$ is a Padé-type approximant to the Fourier series of $f(z_1,z_2)$ with generating system

$$\{\pi_{5,0}, \pi_{5,1}, \pi_{5,2}, \pi_{5,3}, \pi_{5,4}, \pi_{5,5}\},$$

or simply a Padé-type approximant to $f(z_1, z_2)$.

Example 3.5.4. For any $a \in \mathbb{R}$, let f_a be the real-valued function

$$f_a(t) = \sinh(at) \quad (t \in \mathbb{R}).$$

The Fourier series $F_a(t)$ of f_a into $(-\pi, \pi)$ is given by

$$F_a(t) = \frac{2\sinh(a\pi)}{\pi} \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}\nu}{\nu^2 + a^2} \sin(\nu t) = \sum_{\substack{\nu=-\infty \\ (\nu \neq 0)}}^{\infty} \frac{(-1)^{\nu}\, i\, \nu \sin(a\pi)}{\pi\,(\nu^2 + a^2)} e^{i\nu t}\ .$$

Consider the $\mathbb{C}$-linear functional T_{f_a} associated with f_a:

$$T_{f_a} : \mathbb{P}(\mathbb{C}) \to \mathbb{C} : x^{\nu} \mapsto T_{f_a}(x^{\nu}) = c_{\nu}^{(a)} = \begin{cases} 0, & \text{if} \quad \nu = 0 \\ \dfrac{(-1)^{\nu}\, i\, \nu \sinh(a\pi)}{\pi\,(\nu^2 + a^2)}, & \text{if} \quad \nu = 1,2,\ldots \end{cases}$$

If

$$\mathrm{M} = (\pi_{m,k})_{m \geq 0, 0 \leq k \leq m}$$

is a complex infinite triangular matrix, then, for any $m \geq 0$, a Padé-type approximant to $f_a(t)$ is a function

$$\mathrm{Re}(m/m+1)_{f_a}(t) = 2\ \mathrm{Re}\left[\frac{e^{-it} T_{f_a}\left(\dfrac{V_{m+1}(e^{-it}) - V_{m+1}(x)}{e^{-it} - x}\right)}{V_{m+1}(e^{-it})}\right] \qquad (-\pi < t < \pi),$$

where

$$V_{m+1}(x) = \gamma \prod_{k=0}^{m} (x - \pi_{m,k})$$

is the generating polynomial of this approximation ($\gamma \in \mathbb{R} - \{0\}$) (see[43]).

(a). If $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = \pi_{3,3} = 0$, then

- $V_4(x) = x^4$

and

$$\mathrm{Re}(3/4)_{f_a}(t) = \frac{2.\sinh(a\pi)}{\pi} \sum_{\nu=1}^{3} \frac{(-1)^{\nu+1}\nu}{\nu^2 + a^2} \sin(\nu t) =$$

$$= \frac{2.\sinh(a\pi)}{\pi} \left(\frac{1}{1+a^2} \sin t - \frac{2}{4+a^2} \sin 2t + \frac{3}{9+a^2} \sin 3t \right) \quad ([43]).$$

Hence,

a	$1/2$		2	
t	$f_{1/2}(t)$	$\mathrm{Re}(3/4)_{f_{1/2}}(t)$	$f_2(t)$	$\mathrm{Re}(3/4)_{f_2}(t)$
0	0	0	0	0
$-\frac{\pi}{6}$	-0.2648002	-0.2954575	-1.249367	-19.476309
$\frac{\pi}{5}$	0.3193525	0.308832	1.614488	16.920309
$\frac{\pi}{4}$	0.4028703	0.3025888	2.3012989	9.3066229
$\frac{\pi}{3}$	0.5478534	0.2660742	3.9986913	-7.3807882
$\frac{\pi}{2}$	0.8686709	0.4436537	11.548739	-5.2446673

At first glance, the numerical results in the above table show the insufficiency of our choice (at least for $a = 2$). However, for m enough large and

$$\pi_{m,0} = \ldots = \pi_{m,m} = 0\,,$$

the corresponding Padé-type approximation coincides with the partial sum of the first m terms of $F_a(t)$ and therefore his effectiveness is given and encouraging.

(b). If $m = 3$ and $\pi_{3,0} = \pi_{3,1} = \pi_{3,2} = 0$, $\pi_{3,3} = -\frac{2}{3}$, then

- $V_4(x) = x^4 + \frac{2}{3}x^3$,

- $$e^{-it}T_{f_a}\left[\frac{V_4(e^{-it}) - V_4(x)}{e^{-it} - x}\right] = e^{-it}T_{f_a}\left(e^{-3it} + e^{-2it}x + e^{-it}x^2 + x^3 + \frac{2}{3}e^{-2it} + \frac{2}{3}e^{-it}x + \frac{2}{3}x^2\right)$$

$$= e^{-it}\left(\left[c_0^{(a)}\right]e^{-3it} + \left[c_1^{(a)} + \frac{2}{3}c_0^{(a)}\right]e^{-2it} + \left[c_2^{(a)} + \frac{2}{3}c_1^{(a)}\right]e^{-it} + \left[c_3^{(a)} + \frac{2}{3}c_2^{(a)}\right]\right)$$

$$= e^{-4it}\cdot\left(\left[c_1^{(a)}\right]e^{it} + \left[c_2^{(a)} + \frac{2}{3}c_1^{(a)}\right]e^{2it} + \left[c_3^{(a)} + \frac{2}{3}c_2^{(a)}\right]e^{3it}\right),$$

- $V_4\left(e^{-it}\right) = e^{-4it}\left(1 + \frac{2}{3}e^{it}\right)$.

Thus,

$$\mathrm{Re}(3/4)_{f_a}(t) = 2\ \mathrm{Re}\frac{\left(\left[c_1^{(a)}\right]e^{it} + \left[c_2^{(a)} + \frac{2}{3}c_1^{(a)}\right]e^{2it} + \left[c_3^{(a)} + \frac{2}{3}c_2^{(a)}\right]e^{3it}\right)}{1 + \frac{2}{3}e^{it}}$$

$$= \frac{2\sinh(a\pi)}{\pi\,(13+12\cos t)}\left(\left[\frac{13}{1+a^2}-\frac{12}{4+a^2}\right]\sin t+\left[\frac{6}{1+a^2}-\frac{26}{4+a^2}+\frac{18}{9+a^2}\right]\sin 2t\right.$$

$$\left.+\left[\frac{27}{9+a^2}-\frac{12}{4+a^2}\right]\sin 3t\right).$$

It follows that (see[43])

a	1/2		2	
t	$f_{1/2}(t)$	$\mathrm{Re}(3/4)_{f_{1/2}}(t)$	$f_2(t)$	$\mathrm{Re}(3/4)_{f_2}(t)$
0	0	0	0	0
$-\frac{\pi}{6}$	-0.2648002	-0.2773082	-1.249367	-2.0063189
$\frac{\pi}{5}$	0.3193525	0.3317165	1.614488	2.1108546
$\frac{\pi}{4}$	0.4028703	0.4127542	2.3012989	2.0641907
$\frac{\pi}{3}$	0.5478534	0.7364422	3.9986913	1.6883143
$\frac{\pi}{2}$	0.8686709	0.8430904	11.548739	3.4292061

These results seem to be enough successful (at least for the case $a=\frac{1}{2}$). But, on the other hand, some unexpected difficulties appear: if, for example, $a=-3$ then

$$\mathrm{Re}(3/4)_{f_{-3}}\left(\frac{\pi}{4}\right)=25.194418 \qquad \text{while} \qquad f_{-3}\left(\frac{\pi}{4}\right)=-5.2279719\,,$$

$$\mathrm{Re}(3/4)_{f_{-3}}\left(\frac{\pi}{3}\right)=2.0744719 \qquad \text{while} \qquad f_{-3}\left(\frac{\pi}{3}\right)=-11.548739\,,$$

etc.

Obviously, the variation of the real parameter a may cause spectacular perturbations in the behavior of our approximants and, therefore, we must seek for a more satisfactory approximation in the generalized Padé-type sense.

(c). Assume that $|a| < A < \infty$ and consider the analytic function

$$f:\Delta^1(0,A)\times\Delta^1(0,\pi)\to\mathbb{C}:(z_1,z_2)\mapsto f(z_1z_2):=\sinh(z_1,z_2).$$

It is clear that $f\in OL^2(\Delta^1(0,A)\times\Delta^1(0,\pi))$ and $f(a,t)=f_a(t)$ for every $a\in(-A,A)$ and $t\in(-\pi,\pi)$.

As it showed in *Example 3.5.3.(e)*, an orthonormal basis for $OL^2(\Delta^1(0,A)\times\Delta^1(0,\pi))$ is the set of monomials

$$\left\{\varphi_{j_1,j_2}(z_1,z_2)=\frac{\sqrt{j_1+1}\sqrt{j_2+1}}{\sqrt{A\pi\,A^{j_1+1}\pi^{j_2+2}}}\,z_1^{j_1}z_2^{j_2}\;:\;j_1=0,1,2,\ldots,j_2=0,1,2,\ldots\right\}.$$

The elements of this basis can be easily arranged in a sequence as follows:

$\varphi_0:=\varphi_{0,0},\varphi_1:=\varphi_{1,0},\varphi_2:=\varphi_{0,1},\varphi_3:=\varphi_{2,0},\varphi_4:=\varphi_{1,1},\varphi_5:=\varphi_{0,2},\varphi_6:=\varphi_{3,0},\varphi_7:=\varphi_{2,1},\ldots$

With this arrangement, the first six elements of this sequence are

$$\varphi_0(z_1,z_2)=A^{-\frac{3}{2}}\pi^{-\frac{5}{2}}\,,$$

$$\varphi_1(z_1,z_2)=\sqrt{2}A^{-\frac{5}{2}}\pi^{-\frac{5}{2}}z_1\,,$$

$$\varphi_2(z_1,z_2)=\sqrt{2}A^{-\frac{3}{2}}\pi^{-\frac{7}{2}}z_2\ ,$$

$$\varphi_3(z_1,z_2)=\sqrt{3}A^{-\frac{7}{2}}\pi^{-\frac{5}{2}}z_1^2\ ,$$

$$\varphi_4(z_1,z_2)=2\,A^{-\frac{5}{2}}\pi^{-\frac{7}{2}}z_1z_2\ ,$$

$$\varphi_5(z_1,z_2)=\sqrt{3}A^{-\frac{3}{2}}\pi^{-\frac{9}{2}}z_2^2\ .$$

Further, the Fourier coefficients of f with respect to this basis are the successive elements of the set

$$\left\{a_{j_1,j_2}^{(f)}=\frac{1-(-1)^{j_1}}{2\ j_1!\ j_2!}\frac{\sqrt{A\pi}\ A^{j_1+1}\ \pi^{j_2+1}}{\sqrt{j_1+1}\ \sqrt{j_2+1}}:j_1=0,1,2,\ldots,j_2=0,1,2,\ldots\right\}.$$

These coefficients can be arranged in a sequence

$$a_0^{(f)}:=a_{0,0}^{(f)},a_1^{(f)}:=a_{1,0}^{(f)},a_2^{(f)}:=a_{0,1}^{(f)},a_3^{(f)}:=a_{2,0}^{(f)},a_4^{(f)}:=a_{1,1}^{(f)},a_5^{(f)}:=a_{0,2}^{(f)},$$

$$a_6^{(f)}:=a_{3,0}^{(f)}\ldots$$

The first six elements of this sequence are the numbers

$$a_0^{(f)}=0\,,$$

$$a_1^{(f)}=\frac{A^{\frac{5}{2}}\pi^{\frac{5}{2}}}{\sqrt{2}},$$

$$a_2^{(f)}=0\,,$$

$$a_3^{(f)}=0\,,$$

$$a_4^{(f)} = \frac{A^{\frac{5}{2}}\pi^{\frac{7}{2}}}{2},$$

$$a_5^{(f)} = 0.$$

For any $m \geq 0$, choose a finite set of pair-wise distinct points

$$\left\{\pi_{m,0} = \left(\pi_{m,0}^{(1)}, \pi_{m,0}^{(2)}\right), \pi_{m,1} = \left(\pi_{m,1}^{(1)}, \pi_{m,1}^{(2)}\right), \ldots, \pi_{m,m} = \left(\pi_{m,m}^{(1)}, \pi_{m,m}^{(2)}\right)\right\} \subset \Delta^1(0,A)\times\Delta^1(0,\pi),$$

in such a way that the determinant

$$\det\left[\overline{\varphi_j\left(\pi_{m,k}\right)}\right]_{k,j} = \begin{vmatrix} \overline{\varphi_0\left(\pi_{m,0}\right)} & \overline{\varphi_1\left(\pi_{m,0}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,0}\right)} \\ \overline{\varphi_0\left(\pi_{m,1}\right)} & \overline{\varphi_1\left(\pi_{m,1}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,1}\right)} \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \overline{\varphi_0\left(\pi_{m,m}\right)} & \overline{\varphi_1\left(\pi_{m,m}\right)} & \cdots & \overline{\varphi_m\left(\pi_{m,m}\right)} \end{vmatrix}$$

is different from zero.

If $j \leq m$, we set

$$c_j^{(m)}(z_1,z_2) := \sum_{k=0}^{m} \frac{K_{\Delta^1(0,A)\times\Delta^1(0,\pi)}((z_1,z_2),(\pi_{m,k}^{(1)},\pi_{m,k}^{(2)}))}{\overline{\varphi_j(\pi_{m,k})}}$$

$$= \sum_{k=0}^{m} \frac{A}{\pi\left(A^2 - z_1\overline{\pi_{m,k}^{(1)}}\right)^2\left(\pi^2 - z_2\overline{\pi_{m,k}^{(2)}}\right)^2\overline{\varphi_j\left(\pi_{m,k}\right)}}$$

where $K_{\Delta^1(0,A)\times\Delta^1(0,\pi)}(\cdot,\cdot)$ is the Bergman kernel function into $\Delta^1(0,A)\times\Delta^1(0,\pi)$. The sum

$$\left(GPTA/m\right)_f(z_1,z_2) := \sum_{j=0}^{m} a_j^{(f)}\, c_j^{(m)}(z_1,z_2)$$

$$= \frac{A}{\pi}\sum_{j=0}^{m} a_j^{(f)} \sum_{k=0}^{m} \frac{1}{\left(A^2 - z_1.\overline{\pi_{m,k}^{(1)}}\right)^2\left(\pi^2 - z_2.\overline{\pi_{m,k}^{(2)}}\right)^2\overline{\varphi_j\left(\pi_{m,k}\right)}}$$

is then a generalized Padé-type approximant to the complex-valued function

$f(z_1,z_2):=\sinh(z_1 z_2)$ into the bounded open set $\Delta^1(0,A)\times\Delta^1(0,\pi)\subset\mathbf{C}^2$.

If, for instance, $m=2$, then there holds

$$(GPTA/m)_f(z_1,z_2)$$

$$=\frac{A^6\pi^4}{2}\left\{\frac{1}{\left(A^2-z_1\overline{\pi_{2,0}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{2,0}^{(2)}}\right)^2\overline{\pi_{2,0}^{(1)}}}+\frac{1}{\left(A^2-z_1\overline{\pi_{2,1}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{2,1}^{(2)}}\right)^2\overline{\pi_{2,1}^{(1)}}}\right.$$

$$\left.+\frac{1}{\left(A^2-z_1\overline{\pi_{2,2}^{(1)}}\right)^2\left(\pi^2-z_2\overline{\pi_{2,2}^{(2)}}\right)^2\overline{\pi_{2,2}^{(1)}}}\right\},$$

under the constraint:

$$\det\begin{pmatrix}1 & \pi_{2,0}^{(1)} & \pi_{2,0}^{(2)}\\ 1 & \pi_{2,1}^{(1)} & \pi_{2,1}^{(2)}\\ 1 & \pi_{2,2}^{(1)} & \pi_{2,1}^{(2)}\end{pmatrix}\neq 0.$$

Thus, for $A=5$ and

$$\pi_{2,0}^{(1)}=\frac{1}{6},\qquad \pi_{2,0}^{(2)}=\frac{1}{5},$$
$$\pi_{2,1}^{(1)}=-\frac{1}{4},\qquad \pi_{2,1}^{(2)}=\frac{1}{3},$$
$$\pi_{2,2}^{(1)}=-\frac{1}{2},\qquad \pi_{2,2}^{(2)}=-1{,}1957901,$$

a generalized Padé-type approximant to $f(z_1,z_2)$ is given by the expression

$$(GPTA/2)_f(z_1,z_2)=761008.52$$

$$\left\{\frac{5400}{(150-z_1)^2(49.348022-z_2)^2}-\frac{576}{(100+z_1)^2(29.608813-z_2)^2}\right.$$

$$\left.-\frac{8}{(50+z_1)^2(9.8696044+1.1957901\,z_2)^2}\right\}$$

and indicatively we have

$$(GPTA/2)_f\left(-3,\frac{\pi}{4}\right)=-5.2279763\text{, while}\quad f\left(-3,\frac{\pi}{4}\right)=f_{-3}\left(\frac{\pi}{4}\right)=5.2279719$$

and

$$(GPTA/2)_f\left(-3,\frac{\pi}{3}\right)=-4.1422386\text{, while}\quad f\left(-3,\frac{\pi}{3}\right)=f_{-3}\left(\frac{\pi}{3}\right)=11.548739\,.$$

Also, for A very small and such that

$$\max\left\{\left|\pi_{2,0}^{(1)}\right|,\left|\pi_{2,1}^{(1)}\right|,\left|\pi_{2,2}^{(1)}\right|\right\}< A\,,$$

if

$$\max\left\{\left|\pi_{2,0}^{(2)}\right|,\left|\pi_{2,1}^{(2)}\right|,\left|\pi_{2,2}^{(2)}\right|\right\}< \pi\,,$$

one has

$$(GPTA/2)_f(0,t)\approx f(0,t)=f_0(t),\qquad \text{for any } t\in(-\pi,\pi).$$

However and in spite of these promising results, the best choice for the interpolation points

$$\pi_{m,0},\pi_{m,1},\ldots,\pi_{m,m}\in\Delta^1(0,A)\times\Delta^1(0,\pi)$$

remains a central, open and difficult optimization problem of several complex variables.

Open Questions

The Theory of Padé and Padé-type approximation to Fourier series has developed in various directions and presently is far from complete. In this *Section*, I have included a collection of general open problems in the hope that this may be one way to get them solved. Some of these look simple. The fact that they are unsolved shows quite clearly that we have barely begun to understand what really goes on this area of Approximation Theory, in spite of progress that has been made.

In the one variable setting, *the algebraic properties of the approximants remain to be studied*, as well as *the existence and determination of (feasible) best interpolation points*.

The algorithmic part should be similarly *used.* Some generalizations are also of interest, the most important of which seems to be the non-periodic case.

The development of powerhouse techniques like integral representations managed to cut us off from the roots of rational approximation. However, *consideration of* (*composed*) *Padé-type operators may lead to considerable functional analytic questions*. For example, *given any operator*

$$\Lambda : L^2(C) \to L^2(C),$$

does there exist an infinite triangular interpolation matrix such that the corresponding sequence of (*composed*) *Padé-type operators converges to* Λ *with respect to some topology?* (see *Chapter 2*, *Section 2.2.*) We also emphasize to *the theoretical and practical importance of the Padé-type approximation to the study of integral equations, by means of Padé-type approximate equations* (see *Introduction* of *Chapter 2*, page 165).

In the multidimensional case, there are several open theoretical problems.

First, *a complete knowledge of the Bergman kernel for an open bounded set* $\Omega \subset \mathbb{C}^n$ *is indispensable for a detailed description of generalized Padé-type approximants to analytic* L^2 *−functions in* Ω*. The best choice of the orthonormal basis for* $OL^2(\Omega)$ *and of the generating systems in* Ω *is* also *a general and difficult problem.* Further, in analogy to the one variable case, a fundamental functional analytic question is to know *if, for every operator*

$$\Lambda : OL^2(\Omega) \to OL^2(\Omega),$$

one can find an orthonormal basis for $OL^2(\Omega)$ *and generating systems in* Ω *such that the corresponding sequence of generalized Padé-type operators for* $OL^2(\Omega)$ *converges to* Λ *with respect to some topology.*

Second, *the extension and study of classical or* L^p *−Markov's inequalities into more general families of compact subsets of* $\mathbb{C}^n$ *is the component key for a comprehensive universal discussion of generalized Padé-type approximation to continuous functions on compact sets* $E \subset \mathbb{C}^n$. Moreover, *the choice of a polynomial self-summable orthonormal basis for* $L^2(E,\mu)$ *and of generating systems may lead to better approximations: their best choice is an important and difficult question.*

On the other hand, *the introduction of generalized Padé-type operators for* $C^\infty(E)$ *would* probably *be a useful tool in the approximate study of some special operators*

$$C^\infty(E) \to C^\infty(E).$$

Finally, *the algebraic properties of these approximants and operators and their algorithmic aspects should be studied.*

As it is pointed out, all these ideas and methods can be extended into abstract functional Hilbert spaces (*Section 3.4*). *The existence of a self-summable countable orthonormal basis in a functional Hilbert space H and of generating systems leading to the best generalized Padé-type* *approximation to elements to H is the main general question.* The standard functional problem of *finding convenient and efficient generalized Padé-type approximants to a given operator*

$$F : H \to H$$

is another interesting problem.

The application of the above Theory to other branches of Mathematics is also an open question and probably the most fascinating, since the close interplay between the abstract and the concrete is the most useful aspect and the main criterion for obtaining new and research problems.

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